Lecture 2: Extensive Form Games: Refinements

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Today’s Lecture

- In dynamic games, NE may not be strong enough
- Example: an entry game
- Refining the NE: Subgame Perfect Equilibrium (SPNE)
- Refining the NE: Weak Perfect Bayesian (or Weak Sequential) Equilibrium
- Problem: may not be SPNE
- Refining WPBE and SPNE: Perfect Bayesian Equilibrium
- Refining PBE: Sequential Equilibrium
- Trembling-hand perfection in extensive-form games (Lecture 2b)
Example 1

- An dynamic game, the "entry game":

```
  E
 /  \
\2/   \
0 2

F
A
I

E
/
F
A
I

-3 -1
-2 -1
  1  3
  2  1
```
Example 1 (continued)

- Normal form:

<table>
<thead>
<tr>
<th>E</th>
<th>A if In</th>
<th>F if In</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out, A if In</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>Out, F if In</td>
<td>0,2</td>
<td>0,2</td>
</tr>
<tr>
<td>In, A if In</td>
<td>3,1</td>
<td>-2,-1</td>
</tr>
<tr>
<td>In, F if In</td>
<td>1,-2</td>
<td>-3,-1</td>
</tr>
</tbody>
</table>

- Nash equilibria:

- Are all of them reasonable? Problem of credibility

- Shouldn’t prediction satisfy sequential rationality? Yes! Need to refine Nash Equilibrium concept
Subgame Perfect Nash Equilibrium

• Definition: A subgame of $\Gamma_E$ satisfies:
  
  a) Begins at an info set with one node
  
  b) If a node of an info set is in the subgame, then all of the others also are

• Example 1: whole subgame and subgame starting at E decision node

• Definition: A strategy profile $(\sigma_1, ..., \sigma_I)$ in $\Gamma_E$ is a subgame perfect Nash equilibrium if it induces a NE in every subgame

• By definition every SPNE is a NE (method 1 to find SPNE: select among NE)

• In games of perfect information, SPNE is equal to the set of NE derived by backwards induction

• More generally, SPNE can be found by finding NE in every subgame and substituting backwards (method 2 to find SPNE)
Example 2
• Normal form:

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>I</th>
<th>F</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out</td>
<td>0, 2</td>
<td>0,2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>InS</td>
<td>-1,-1</td>
<td>3,0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>InW</td>
<td>-1,-1</td>
<td>2,1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

• NE: (Out, F), (InS, A). SPNE: . Plausible?

• Sequentially rationality principle: play should be optimal for some belief (about type of entry)

• Idea of WPBE: define beliefs and apply sequential rationality
Formal Definitions: Beliefs

- Remember that:
  $\chi$ is the set of nodes
  $s(x)$ are the set of successors of node $x$
  $\Xi$ is the collection of information sets (partition of $\chi$)
  $H(x)$ is the information set containing node $x$
  $\bigcup_{x \in H}s(x)$ all the successors at information set $H$
  $\iota(H)$ determines the playing player at info set $H$

- Definition: A system of beliefs $\mu$ is a mapping $\mu : \chi \to [0, 1]$ such that $\sum_{x \in H} \mu(x) = 1$ for any $H \in \Xi$.

- In previous example: (1) in $H_1$ and $\left(\frac{2}{3}, \frac{1}{3}\right)$ in $H_2$.

- Interpretation: probability assessment of likelihood of each node at each info set
Sequential Rationality: Restrictions on Strategies

- Definition: A behaviour strategy profile $\delta$ is \textit{sequentially rational} given a system of beliefs $\mu$ if, for all $H$, the actions of $\iota(H)$ at $H \cup (\bigcup_{x \in HS(x)} x)$ are optimal given an initial probability over $H$ defined by $\mu$, and given that all the other players adhere to $\delta_{-i}$.

- Given beliefs $[1, \left(\frac{2}{3}, \frac{1}{3}\right)]$, the only sequentially rational strategy profile is $(InS, A)$
  a) at $H_2$ (I playing), the payoff from $A$ is $\frac{2}{3} \times 0 + \frac{1}{3} \times 1 = \frac{1}{3}$
  whereas the payoff from $F$ is $\frac{2}{3} \times (-1) + \frac{1}{3} \times (-1) = -1$
  b) at $H_1$ (E playing and given $\delta_{-i} = A$),
  the payoff from $Out$ is 0, from $InS$ is 3 and from $InW$ is 2

- Can you see why, given beliefs $[1, \left(\frac{2}{3}, \frac{1}{3}\right)]$, $(Out, F)$ is not sequentially rational?

- In fact $F$ is never going to be a sequentially rational strategy (for any belief)!

- See MWG book for a definition in pure and mixed strategies
Restrictions on Beliefs

- Beliefs should be "consistent" with the strategy profile (whenever possible)

- Example (1): suppose that E plays \( (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}) \) (=\( (p(Out), p(InS), p(InW)) \))
  
  What should be the beliefs of I on \( H_2 \)? Exactly, \( (\frac{2}{3}, \frac{1}{3}) \)

- Example (2): suppose that E plays \( (\frac{1}{2}, \frac{1}{2}, 0) \)
  
  What should be the beliefs of I on \( H_2 \) now? \( (1, 0) \)

- Formally, we use Bayes rule. For a strategy profile \( \delta^* \) and for any \( x \in H \)
  \[
  \mu^*(x) = \frac{\Pr ob(x \mid \delta^*)}{\Pr ob(H \mid \delta^*)}
  \]

- Example (3): suppose that E plays \( (1, 0, 0) \)
  
  What should be the beliefs of I? Bayes rule can’t be used (\( \Pr ob(H \mid \delta^*) = 0 \))
  
  In the WPBE, we will assign one belief (any but one needs to be assigned!)
Weak Perfect Bayesian Equilibrium

- **Definition:** $(\delta^*, \mu^*)$ is a *Weak Perfect Bayesian equilibrium* iff
  a) the behaviour strategy profile $\delta^*$ is sequentially rational given $\mu^*$, and
  b) wherever possible $\mu^*$ is computed from $\delta^*$ using Bayes rule. That is for any information set $H$ such that $\text{Pr}ob(H \mid \delta^*) > 0$, we have that, for any $x \in H$,

$$
\mu^*(x) = \frac{\text{Pr}ob(x \mid \delta^*)}{\text{Pr}ob(H \mid \delta^*)}.
$$

- **Example:** $(\delta^*, \mu^*) = \{[InS, A], [1, (1, 0)]\}$ is the only WPBE

- **Remarks:**
  Nash equilibrium requires sequential rationality only in $H$ s.t. $\text{Pr}ob(H \mid \delta^*) > 0$
  WPBE also requires specification of (any) beliefs and an optimal action in $H$ such that $\text{Pr}ob(H \mid \delta^*) = 0$
• Now I is willing to play $F$ if she knew that $In.S$ has been played
Suppose that $[[p_{Out},p_{InS},p_{InW}),(p_F,p_A)], [1,(\mu_1, \mu_2)]]$ is a WPBE

Given the system of beliefs, the optimal decision at $H_2$ is

Play $F$ iff $\mu_1(-1) + (1 - \mu_1)(-1) \geq \mu_1(-2) + (1 - \mu_1)(1)$ or iff $\mu_1 \geq \frac{2}{3}$

- Suppose that $\mu_1 > \frac{2}{3}$, then the sequentially rational decision at $H_2$ is $p_F = 1$ and therefore $p_A = 0$

  Then, E’s sequentially rational strategy at $H_1$ given the strategy of I is $p_{InW} = 1, p_{Out} = p_{InS} = 0$ since they payoffs are 1, $-1$ and 0, resp.

  Then, by Bayes rule, $\mu_1 = 0$ (and $\mu_2 = 1$). Contradiction!!

- Suppose that $\mu_1 < \frac{2}{3}$, then the sequentially rational decision at $H_2$ is $p_F = 0$ and therefore $p_A = 1$

  Then, E’s sequentially rational strategy at $H_1$ given the strategy of I is $p_{InS} = 1, p_{Out} = p_{InW} = 0$ since they payoffs are 3, 0 and 2, resp.

  Then, by Bayes rule, $\mu_1 = 1$ and $\mu_2 = 0$. Contradiction!!
• Suppose that $\mu_1 = \frac{2}{3}$, then (a) assume that either $p_{InS} > 0$ or $p_{InW} > 0$
Can apply Bayes and since $\mu_1 = \frac{2}{3}$, $p_{InS} = 2p_{InW}$ & $p_{InS} > 0$ and $p_{InW} > 0$
Then player I should be indifferent between $InS$ and $InW$:
$p_F(-1) + (1 - p_F)(3) = p_F(1) + (1 - p_F)(2)$ or $p_F = \frac{1}{3}$ (and $p_A = \frac{2}{3}$)
Then payoffs for E for $InS$ (same for $InW$) are $\frac{1}{3}(-1) + \frac{2}{3}(3) = \frac{5}{3}$ whereas for $Out$ are 0
Therefore $p_{Out} = 0$ and then $p_{InS} = \frac{2}{3}$ and $p_{InW} = \frac{1}{3}$
$[[ (0, \frac{2}{3}, \frac{1}{3} ), (\frac{1}{3}, \frac{2}{3} ) ] , [1, (\frac{2}{3}, \frac{1}{3}) ]]$ is a WPBE

• Suppose that $\mu_1 = \frac{2}{3}$ and (b) assume that $p_{InS} = 0$ and $p_{InW} = 0$ (then $p_{Out} = 1$)
But then, given $(p_F, p_A)$, E’s payoff for playing $Out$ should be, at least, as large as for playing $InW$ (and $InS$ as well)
i.e. $0 \geq p_F(1) + (1 - p_F)(2)$ or $p_F \geq 2$. Contradiction!!!
WPBE may not be SPNE

- Problem: beliefs only need to be consistent in the equilibrium path
- Example 1 again:
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• NE: [(In, A if In), A if In] & [(Out, A or F if In), F if In]. SPNE: the first

• May the second be supported as a WPNE? If the beliefs [1, 1, (1, 0)] are added:
  Consistent beliefs when possible? Yes, because last info set is not reached in eq.
  Sequentially rational given beliefs and other’s strategy? Yes, because...
  at the last info set, given (1, 0) beliefs, F is sequentially rational
  at the before-to-last node, given I’s strategy (F), A is sequentially rational
  at the first node, given I’s strategy (F), (Out, A if In) is sequentially rational
Perfect Bayesian Equilibrium

- Definition: A *Perfect Bayesian Equilibrium* (PBE) is a WPBE that induces a WPBE in every subgame.

- Previous WPBE: \([(Out, A \text{ if } In), F \text{ if } In], [1, 1, (1, 0)]\) is not a PBE. At the subgame starting at the 2nd node (\([(A \text{ if } In), F \text{ if } In], [1, (1, 0)]\)) is not a WPBE since beliefs at the last info set are not consistent with strategy profile.

- But, PBE still does not place much restrictions on out-of-equilibrium beliefs (see next slide).
Example 4: PBE may still not be strong enough
The strategy profile and the specified beliefs form a WPBE:
Beliefs are consistent (when possible): at \( H_1 \) they are and at \( H_2 \) they are free.
Given beliefs at \( H_2 \), sequentially rational at \( H_2 \) since
\[
0.9 \times 3 + 0.1 \times 5 = 3.2 > 2.8 = 0.9 \times 2 + 0.1 \times 10
\]

Given beliefs at \( H_1 \) and 2’s strategy, sequentially rational at \( H_1 \) since
\[
0.5 \times 2 + 0.5 \times 2 = 2 > 0 = 0.5 \times 0 + 0.5 \times 0
\]

Given that the only subgame is the whole game, this WPBE is also a PBE.

Are player 2 beliefs "sensible"?

Given that Nature assigns 0.5 to each branch and 1’s choice cannot depend on Nature’s outcome, the only sensible beliefs for 2 seem to be \((0.5, 0.5)\).
Sequential Equilibrium

- Idea: make out-of-equilibrium beliefs consistent with strategies that reach all info sets and aren’t very different from equilibrium strategies
- Definition: A behaviour strategy is strictly mixed if every action at each info set is selected with strictly positive probability
- Note: If all the strategies of the strategy profile are strictly mixed, then all info sets are reached and Bayes rule can always be used
- Definition: \((\delta, \mu)\) is consistent iff there exists a sequence of strictly mixed behavior strategy profiles \(\delta_n \to \delta\) such that \(\mu_n \to \mu\), where \(\mu_n\) is derived from \(\delta_n\) using Bayes rule
- Example: in the previous graph, the pair of strategy and beliefs was not consistent Beliefs \((0.5, 0.5)\) at \(H_2\) would have been consistent with the strategy
- Definition: \((\delta^*, \mu^*)\) is a sequential equilibrium (SE) iff \(\delta^*\) is sequentially rational given \(\mu^*\) and \((\delta^*, \mu^*)\) is consistent
• Remark 1: SE places additional restrictions on the beliefs of a WPBE

• Remark 2: any SE is subgame perfect

• Note:

\[ SE \subseteq PBE \subseteq \left\{ \begin{array}{c} WPBE = WSE \\ SPNE \end{array} \right\} \subseteq BNE \]

• Exercise: check that \([(Out, A \text{ if } In), F \text{ if } In]\) in Example 1 is not a SE