Lecture 2: Extensive Form Games: Refinements

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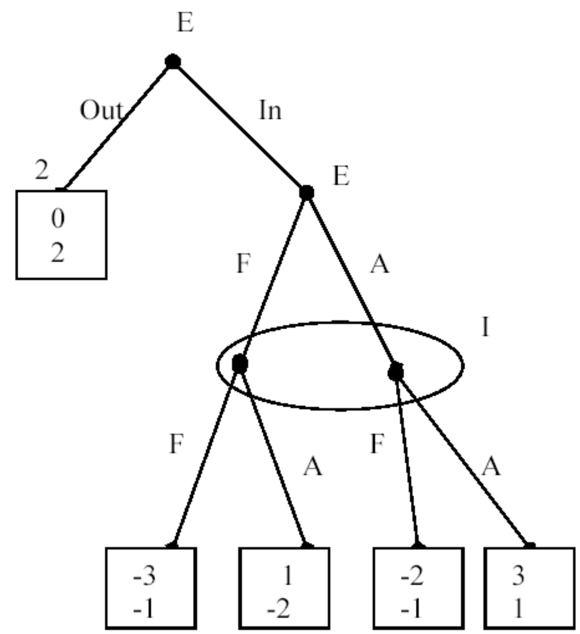
Today's Lecture

- In dynamic games, NE may not be strong enough
- Example: an entry game
- Refining the NE: Subgame Perfect Equilibrium (SPNE)
- Refining the NE: Weak Perfect Bayesian (or Weak Sequential) Equilibrium
- Problem: may not be SPNE
- Refining WPBE and SPNE: Perfect Bayesian Equilibrium
- Refining PBE: Sequential Equilibrium
- Trembling-hand perfection in extensive-form games (Lecture 2b)

Example 1

- An dynamic game, the "entry game":



Example 1 (continued)

- Normal form:

E\I	A if In	F if In
Out, A if In	0,2	0,2
Out, F if In	0,2	0,2
In, A if In	3,1	-2,-1
In, F if In	1,-2	-3,-1

- Nash equilibria:

- Are all of them reasonable? Problem of credibility

- Shouldn't prediction satisfy sequential rationality? Yes! Need to refine Nash Equilibrium concept

Subgame Perfect Nash Equilibrium

- Definition: A subgame of Γ_E satisfies:

  a) Begins at an info set with one node

  b) If a node of an info set is in the subgame, then all of the others also are

- Example 1: whole subgame and subgame starting at E decision node

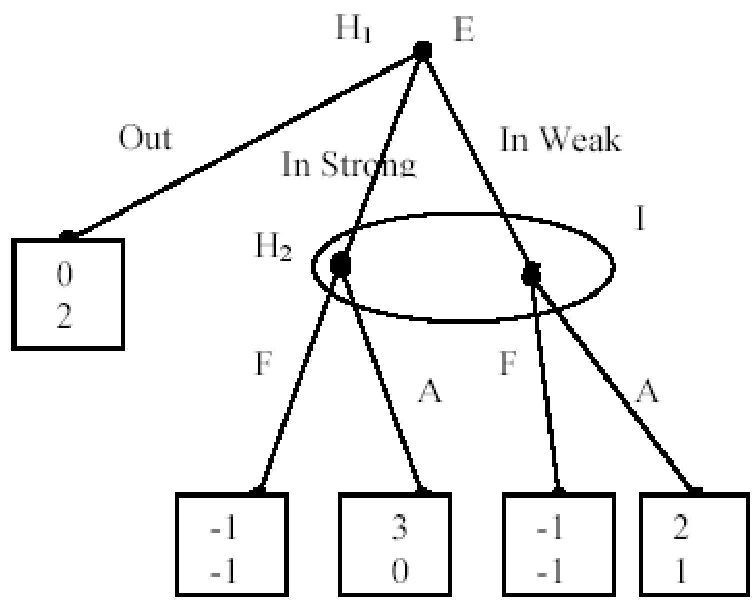
- Definition: A strategy profile $(\sigma_1, ..., \sigma_I)$ in Γ_E is a subgame perfect Nash equilibrium if it induces a NE in every subgame

- By definition every SPNE is a NE (method 1 to find SPNE: select among NE)

- In games of perfect information, SPNE is equal to the set of NE derived by backwards induction

- More generally, SPNE can be found by finding NE in every subgame and substituting backwards (method 2 to find SPNE)

Example 2



- Normal form:

$E\backslash I$	F	A
Out	0, 2	0,2
InS	-1,-1	3,0
InW	-1,-1	2,1

- NE: (Out, F), (InS, A). SPNE: . Plausible?

- Sequentially rationality principle: play should be optimal for some belief (about type of entry)

- Idea of WPBE: define beliefs and apply sequential rationality

Formal Definitions: Beliefs

- Remember that:
  χ is the set of nodes
  $s(x)$ are the set of successors of node x
  Ξ is the collection of information sets (partition of χ)
  $H(x)$ is the information set containing node x
  $\cup_{x\in H} s(x)$ all the successors at information set H
  $\iota(H)$ determines the playing player at info set H

- Definition: A *system of beliefs* μ is a mapping $\mu : \chi \to [0,1]$ such that $\sum_{x\in H} \mu(x) = 1$ for any $H \in \Xi$.

- In previous example: (1) in H_1 and $(\frac{2}{3}, \frac{1}{3})$ in H_2.

- Interpretation: probability assessment of likelihood of each node at each info set

Sequential Rationality: Restrictions on Strategies

- Definition: A behaviour strategy profile δ is *sequentially rational* given a system of beliefs μ if, for all H, the actions of $\iota(H)$ at $H \cup (\cup_{x \in H} s(x))$ are optimal given an initial probability over H defined by μ, and given that all the other players adhere to δ_{-i}.
- Given beliefs $\left[1, \left(\frac{2}{3}, \frac{1}{3}\right)\right]$, the only sequentially rational strategy profile is (InS, A)
  a) at H_2 (I playing), the payoff from A is $\frac{2}{3} * 0 + \frac{1}{3} * 1 = \frac{1}{3}$
  whereas the payoff from F is $\frac{2}{3} * (-1) + \frac{1}{3} * (-1) = -1$
  b) at H_1 (E playing and given $\delta_{-i} = A$),
  the payoff from Out is 0, from InS is 3 and from InW is 2
- Can you see why, given beliefs $\left[1, \left(\frac{2}{3}, \frac{1}{3}\right)\right]$, (Out, F) is not sequentially rational?
- In fact F is never going to be a sequentially rational strategy (for any belief)!
- See MWG book for a definition in pure and mixed strategies

Restrictions on Beliefs

- Beliefs should be "consistent" with the strategy profile (whenever possible)
- Example (1): suppose that E plays $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ (=$(p(Out), p(InS), p(InW))$)
  What should be the beliefs of I on H_2? Exactly, $\left(\frac{2}{3}, \frac{1}{3}\right)$
- Example (2): suppose that E plays $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$
  What should be the beliefs of I on H_2 now? $(1, 0)$
- Formally, we use Bayes rule. For a strategy profile δ^* and for any $x \in H$

$$\mu^*(x) = \frac{\Pr ob(x \mid \delta^*)}{\Pr ob(H \mid \delta^*)}$$

- Example (3): suppose that E plays $(1, 0, 0)$
  What should be the beliefs of I? Bayes rule can't be used $(\Pr ob(H \mid \delta^*) = 0)$
  In the WPBE, we will assign one belief (any but one needs to be assigned!)

Weak Perfect Bayesian Equilibrium

- Definition: (δ^*, μ^*) is a *Weak Perfect Bayesian equilibrium* iff

  a) the behaviour strategy profile δ^* is sequentially rational given μ^*, and

  b) wherever possible μ^* is computed from δ^* using Bayes rule. That is for any information set H such that $\Pr ob(H \mid \delta^*) > 0$, we have that, for any $x \in H$,

$$\mu^*(x) = \frac{\Pr ob(x \mid \delta^*)}{\Pr ob(H \mid \delta^*)}.$$

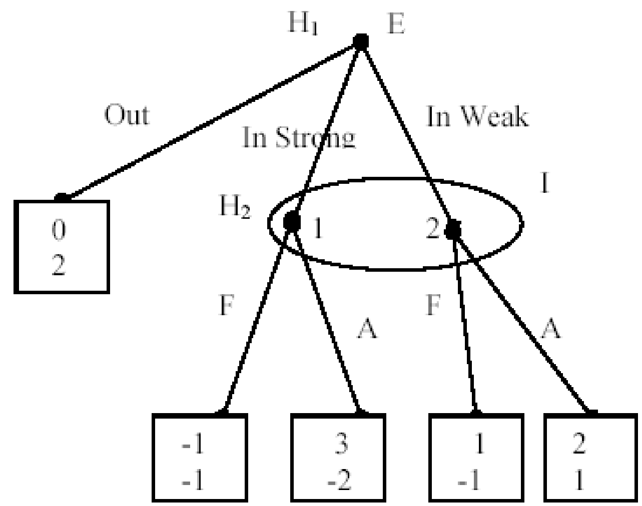
- Example: $(\delta^*, \mu^*) = [[InS, A], [1, (1, 0)]]$ is the only WPBE

- Remarks:

  Nash equilibrium requires sequential rationality only in H s.t. $\Pr ob(H \mid \delta^*) > 0$

  WPBE also requires specification of (any) beliefs and an optimal action in H such that $\Pr ob(H \mid \delta^*) = 0$

Example 3



- Now I is willing to play F if she knew that InS has been played

Suppose that $\left[\left[\left(p_{Out}, p_{InS}, p_{InW}\right), \left(p_F, p_A\right)\right], \left[1, \left(\mu_1, \mu_2\right)\right]\right]$ is a WPBE
Given the system of beliefs, the optimal decision at H_2 is
Play F iff $\mu_1(-1) + (1-\mu_1)(-1) \geq \mu_1(-2) + (1-\mu_1)(1)$ or iff $\mu_1 \geq \frac{2}{3}$

- Suppose that $\mu_1 > \frac{2}{3}$, then the sequentially rational decision at H_2 is $p_F = 1$ and therefore $p_A = 0$
  Then, E's sequentially rational strategy at H_1 given the strategy of I is $p_{InW} = 1, p_{Out} = p_{InS} = 0$ since they payoffs are 1, -1 and 0, resp.
  Then, by Bayes rule, $\mu_1 = 0$ (and $\mu_2 = 1$). Contradiction!!

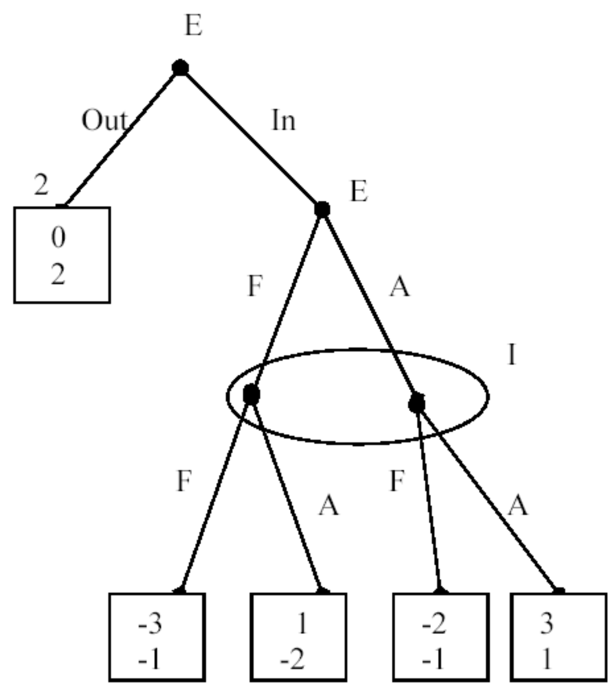
- Suppose that $\mu_1 < \frac{2}{3}$, then the sequentially rational decision at H_2 is $p_F = 0$ and therefore $p_A = 1$
  Then, E's sequentially rational strategy at H_1 given the strategy of I is $p_{InS} = 1, p_{Out} = p_{InW} = 0$ since they payoffs are 3, 0 and 2, resp.
  Then, by Bayes rule, $\mu_1 = 1$ and $\mu_2 = 0$. Contradiction!!

- Suppose that $\mu_1 = \frac{2}{3}$, then (a) assume that either $p_{InS} > 0$ or $p_{InW} > 0$
  Can apply Bayes and since $\mu_1 = \frac{2}{3}$, $p_{InS} = 2p_{InW}$ & $p_{InS} > 0$ and $p_{InW} > 0$
  Then player I should be indifferent between InS and InW:
  $p_F(-1) + (1 - p_F)(3) = p_F(1) + (1 - p_F)(2)$ or $p_F = \frac{1}{3}$ (and $p_A = \frac{2}{3}$)
  Then payoffs for E for InS (same for InW) are $\frac{1}{3}(-1) + \frac{2}{3}(3) = \frac{5}{3}$ whereas for Out are 0
  Therefore $p_{Out} = 0$ and then $p_{InS} = \frac{2}{3}$ and $p_{InW} = \frac{1}{3}$
  $\left[\left[\left(0, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right], \left[1, \left(\frac{2}{3}, \frac{1}{3}\right)\right]\right]$ is a WPBE

- Suppose that $\mu_1 = \frac{2}{3}$ and (b) assume that $p_{InS} = 0$ and $p_{InW} = 0$ (then $p_{Out} = 1$)
  But then, given (p_F, p_A), E's payoff for playing Out should be, at least, as large as for playing InW (and InS as well)
  i.e. $0 \geq p_F(1) + (1 - p_F)(2)$ or $p_F \geq 2$. Contradiction!!!

WPBE may not be SPNE

- Problem: beliefs only need to be consistent in the equilibrium path
- Example 1 again:



- Normal form:

E\I	A if In	F if In
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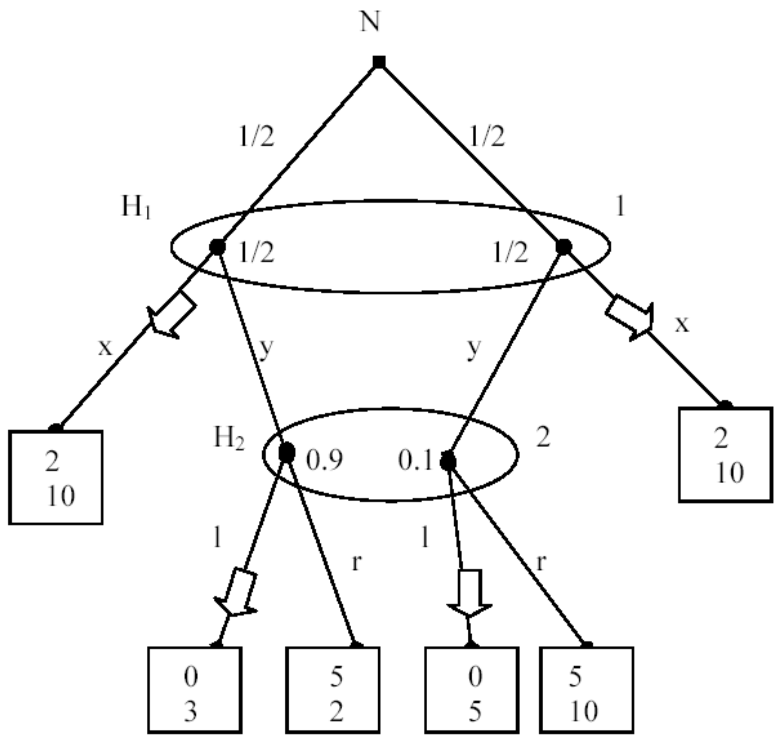
- NE: $[(In, A$ if $In), A$ if $In]$ & $[(Out, A$ or F if $In), F$ if $In]$. SPNE: the first

- May the second be supported as a WPNE? If the beliefs $[1, 1, (1, 0)]$ are added:
  Consistent beliefs when possible? Yes, because last info set is not reached in eq.
  Sequentially rational given beliefs and other's strategy? Yes, because...
  at the last info set, given $(1, 0)$ beliefs, F is sequentially rational
  at the before-to-last node, given I's strategy (F), A is sequentially rational
  at the first node, given I's strategy (F), $(Out, A$ if $In)$ is sequentially rational

Perfect Bayesian Equilibrium

- Definition: A *Perfect Bayesian Equilibrium* (PBE) is a WPBE that induces a WPBE in every subgame
- Previous WPBE: $([(Out, A \text{ if } In), F \text{ if } In], [1, 1, (1, 0)])$ is not a PBE
  At the subgame starting at the 2nd node $([(A \text{ if } In), F \text{ if } In], [1, (1, 0)])$ is not a WPBE since beliefs at the last info set are not consistent with strategy profile
- But, PBE still does not place much restrictions on out-of-equilibrium beliefs (see next slide)

Example 4: PBE may still not be strong enough



- The strategy profile and the specified beliefs form a WPBE:
  Beliefs are consistent (when possible): at H_1 they are and at H_2 they are free
  Given beliefs at H_2, sequentially rational at H_2 since

$$0.9 * 3 + 0.1 * 5 = 3.2 > 2.8 = 0.9 * 2 + 0.1 * 10$$

  Given beliefs at H_1 and 2's strategy, sequentially rational at H_1 since

$$0.5 * 2 + 0.5 * 2 = 2 > 0 = 0.5 * 0 + 0.5 * 0$$

- Given that the only subgame is the whole game, this WPBE is also a PBE
- Are player 2 beliefs "sensible"?
- Given that Nature assigns 0.5 to each branch and 1's choice cannot depend on Nature's outcome, the only sensible beliefs for 2 seem to be $(0.5, 0.5)$

Sequential Equilibrium

- Idea: make out-of-equilibrium beliefs consistent with strategies that reach all info sets and aren't very different from equilibrium strategies
- Definition: A behaviour strategy is *strictly mixed* if every action at each info set is selected with strictly positive probability
- Note: If all the strategies of the strategy profile are strictly mixed, then all info sets are reached and Bayes rule can always be used
- Definition: (δ, μ) is *consistent* iff there exists a sequence of strictly mixed behavior strategy profiles $\delta_n \longrightarrow \delta$ such that $\mu_n \longrightarrow \mu$, where μ_n is derived from δ_n using Bayes rule
- Example: in the previous graph, the pair of strategy and beliefs was not consistent Beliefs $(0.5, 0.5)$ at H_2 would have been consistent with the strategy
- Definition: (δ^*, μ^*) is a *sequential equilibrium* (SE) iff δ^* is sequentially rational given μ^* and (δ^*, μ^*) is consistent

- Remark 1: SE places additional restrictions on the beliefs of a WPBE
- Remark 2: any SE is subgame perfect
- Note:

$$SE \subseteq PBE \subseteq \left\{ \begin{array}{c} WPBE = WSE \\ SPNE \end{array} \right\} \subseteq BNE$$

- Exercise: check that $[(Out, A \text{ if } In), F \text{ if } In]$ in Example 1 is not a SE