Lecture 1: Normal Form Games: Refinements and Correlated Equilibrium

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Today’s Lecture

• Trembling hand perfect equilibrium:
  Motivation, definition and examples

• Proper equilibrium:
  Motivation and examples

• Correlated equilibrium:
  Motivation, definition and examples
Motivation for "Trembling Hands"

- Rationality does not rule out weakly dominated strategies
- In fact, NE can include weakly dominated strategies
- Example: (D,R) in

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>U</td>
<td>1, 1</td>
<td>0, -3</td>
</tr>
<tr>
<td>D</td>
<td>-3, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- But should we expect players to play weakly dominated strategies?
- Players should be completely sure of the choice of the others
- But, what if there is some risk that another player makes a "mistake"?
Trembling Hand Perfection

- For $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$ define...
  for each $i$ and $s_i$, $\varepsilon_i(s_i) \in (0, 1)$;
  $\Delta_\varepsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i) \text{ for all } s_i \in S_i \text{ and } \sum s_i \sigma_i(s_i) = 1\}$;
  "the perturbed game" as $\Gamma_\varepsilon = [I, \{\Delta_\varepsilon(S_i)\}, \{u_i()\}]$

- Interpretation:
  each strategy $s_i$ is played with some minimal probability
  this is the unavoidable probability of a mistake

- A NE $\sigma$ is **trembling hand perfect** if there is some sequence of perturbed games
  $\{\Gamma_{\varepsilon_k}\}_{k=1}^{\infty}$ converging to $\Gamma_N$ for which there is some associated sequence of NE
  $\{\sigma^k\}_{k=1}^{\infty}$ converging to $\sigma$
Alternative Definition and Properties

- Problem: need to compute equilibria of many possible perturbed games

- Proposition: \( \sigma \) is *trembling hand perfect* if and only if there is a sequence of totally mixed strategy profiles \( \sigma^k \) such that \( \sigma^k \rightarrow \sigma \) and, for all \( i \) and \( k \), \( \sigma_i \) is a best response to every \( \sigma^k_{-i} \)

- Counterexample: \((D,R)\) in the previous example

- Corollary: \( \sigma_i \) in a trembling-hand perfect equilibrium cannot be weakly dominated. No weakly dominated pure strategy can be played with positive probability

- Remark: the converse (any NE not involving weakly dominated strategies is trembling hand perfect) is true for two-player games but not for more than two
Existence

- Proposition: Every $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$ with finite strategy sets $S_i$ has a trembling-hand perfect equilibrium

- Corollary: At least there is a NE in which no player plays any weakly dominated strategy with positive probability

- Counterexample: Bertrand-Nash equilibrium allowing for continuous set of prices
Proper Equilibrium

• Example:

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<th>R</th>
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<tbody>
<tr>
<td><strong>U</strong></td>
<td>1, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>0, 0</td>
<td>0, 0</td>
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</table>

- NE: (U,L), (D,R). THP: (U,L) but not (D,R) (weakly dominated)

• But adding two weakly dominated strategies:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>1, 1</td>
<td>0, 0</td>
<td>-9, -9</td>
</tr>
<tr>
<td><strong>M</strong></td>
<td>0, 0</td>
<td>0, 0</td>
<td>-7, -7</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>-9,-9</td>
<td>-7,-7</td>
<td>-7, -7</td>
</tr>
</tbody>
</table>

- NE: (U,L), (M,M), (D,R). THP: (U,L), (M,M)
• (M,M) is THP:
  e.g. consider the totally mixed \((\varepsilon, 1 - 2\varepsilon, \varepsilon)\) for both
  for player 1 (or 2), deviating to \(U\) (or \(L\)): \((\varepsilon - 9\varepsilon) - (-7\varepsilon) = -\varepsilon < 0\)

• Idea of proper equilibrium: more likely to tremble to better strategies:
  second-best actions assigned at most \(\varepsilon\) times the probability of third-best actions,
  fourth-best actions assigned at most \(\varepsilon\) times the probability of third-best actions,
  etc.

• (M,M) is not proper equilibrium:
  e.g. if player 2 puts weight \(\varepsilon\) on \(L\) and \(\varepsilon^2\) on \(R\)
  for player 1, deviating to \(U\): \((\varepsilon - 9\varepsilon^2) - (-7\varepsilon^2) > 0\) for \(\varepsilon\) small

• See Fudenberg and Tirole for a formal definition and properties
Towards Correlated Equilibria

• Example:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
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<tbody>
<tr>
<td>$A$</td>
<td>9, 9</td>
<td>6, 10</td>
</tr>
<tr>
<td>$B$</td>
<td>10, 6</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

• Pure and mixed strategy NE:
  - $(A, b)$ with payoffs $(6, 10)$
  - $(B, a)$ with payoffs $(10, 6)$
  - $\left[\left(\frac{6}{7}, \frac{1}{7}\right), \left(\frac{6}{7}, \frac{1}{7}\right)\right]$ with expected payoffs $\approx (8.57, 8.57)$
New Potential Agreement (1)

- First potential agreement:
  appoint a third party to flip a coin and announce "H" or "T"
  play \((A, b)\) if \(H\) and \((B, a)\) if \(T\)
  expected payoffs \((8, 8)\)

- Are these "mutual best responses"?
  If "H", player 1 knows that player 2 plays \(b\), \(A\) is a best response
  If "T", player 1 knows that player 2 plays \(a\), \(B\) is a best response
same for player 2 (game is symmetric)

- Expected payoffs could dominate mixed NE (e.g. change 9s for 8s)
New Potential Agreement (2)

- Second potential agreement:
  ask third party to roll a dice (number $n$ not observed by players) and
  announce player 1 whether $n$ is in $\{1,2\}$ or in $\{3,4,5,6\}$ and
  announce player 2 whether $n$ is in $\{1,2,3,4\}$ or in $\{5,6\}$
  player 1 plays $B$ if $\{1,2\}$ and $A$ if $\{3,4,5,6\}$
  player 2 plays $a$ if $\{1,2,3,4\}$ and $b$ if $\{5,6\}$
  expected payoffs $(8.33, 8.33)$ (still lower than in mixed, but wait...)

- Are these mutual best responses? e.g. for player 1...
  if $n$ is in $\{1,2\}$, she knows that 2 plays $a$ and then $B$ is a best-response
  if $n$ is in $\{3,4,5,6\}$, she gives $\frac{1}{2}$ to both $a$ and $b$, and then $A$ is a best-response
Correlated Equilibrium: Definition

- Definition in a finite game $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$: 
  $\sigma^*$, a probability distribution over $S_1 \times ... \times S_I$, is a correlated equilibrium iff for all $i$ and for all $s_i$ chosen with positive probability, $s_i$ solves

  $$\max_{s_i} E_{s_{-i}} \left[ u_i(s_i', s_{-i}) \mid s_i, \sigma^* \right]$$

- In the previous second potential agreement, the probability distribution was...

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<th>$b$</th>
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<tbody>
<tr>
<td>$A$</td>
<td>$1/3$</td>
<td>$1/3$</td>
</tr>
<tr>
<td>$B$</td>
<td>$1/3$</td>
<td>$0$</td>
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Example (continued)

- More generally, consider the family:

<table>
<thead>
<tr>
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<th>$a$</th>
<th>$b$</th>
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<tbody>
<tr>
<td>$A$</td>
<td>$\gamma$</td>
<td>$(1-\gamma)/2$</td>
</tr>
<tr>
<td>$B$</td>
<td>$(1-\gamma)/2$</td>
<td>0</td>
</tr>
</tbody>
</table>

- Is it a correlated equilibrium? e.g. for player 1...
  when told to play $B$, she knows that 2 plays $a$. $B$ is a best response
  when told to play $A$, prob of $a$ is $\frac{\gamma}{\gamma+(1-\gamma)/2} = \frac{2\gamma}{1+\gamma}$. $A$ is a best response iff

  $$9 \left( \frac{2\gamma}{1+\gamma} \right) + 6 \left( \frac{1-\gamma}{1+\gamma} \right) \geq 10 \left( \frac{2\gamma}{1+\gamma} \right) + 0 \left( \frac{1-\gamma}{1+\gamma} \right)$$

  or $\gamma \in [0, 3/4]$

- Remarks:
  $\gamma = 0$ corresponds to the previous first potential agreement
  $\gamma = 3/4$ has payoffs $(8.75, 8.75)$, dominating the mixed NE
Mixed NE and Correlated Equilibrium

- Interpretation of a mixed strategy equilibrium:
  players’ randomisations are independent
  condition decisions on private and independent signals
  e.g. \((1/2, 1/2)\) in matching pennies: choose H if the first step of the day is with your right foot and T if it is with your left one

- Interpretation of correlated equilibrium:
  players’ randomisations may be correlated
  decisions may also be conditioned on a public signal
  e.g. realisation of a flip of a coin in the previous first potential agreement