Lecture 1: Normal Form Games: Refinements and Correlated Equilibrium

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April 2006

Today's Lecture

- Trembling hand perfect equilibrium:
  Motivation, definition and examples
- Proper equilibrium:
  Motivation and examples
- Correlated equilibrium:
  Motivation, definition and examples

Motivation for "Trembling Hands"

- Rationality does not rule out weakly dominated strategies
- In fact, NE can include weakly dominated strategies
- Example: (D,R) in

	L	R
U	1 , 1	0,-3
D	-3 , 0	0, 0

- But should we expect players to play weakly dominated strategies?
- Players should be completely sure of the choice of the others
- But, what if there is some risk that another player makes a "mistake"?

Trembling Hand Perfection

- For $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$ define...
  for each i and s_i, $\varepsilon_i(s_i) \in (0,1)$;
  $\Delta_\varepsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i)$ for all $s_i \in S_i$ and $\Sigma_{s_i}\sigma_i(s_i) = 1\}$;
  "the perturbed game" as $\Gamma_\varepsilon = [I, \{\Delta_\varepsilon(S_i)\}, \{u_i()\}]$

- Interpretation:
  each strategy s_i is played with some minimal probability
  this is the unavoidable probability of a mistake

- A NE σ is *trembling hand perfect* if there is some sequence of perturbed games $\{\Gamma_{\varepsilon^k}\}_{k=1}^{\infty}$ converging to Γ_N for which there is some associated sequence of NE $\{\sigma^k\}_{k=1}^{\infty}$ converging to σ

Alternative Definition and Properties

- Problem: need to compute equilibria of many possible perturbed games

- Proposition: σ is *trembling hand perfect* if and only if there is a sequence of totally mixed strategy profiles σ^k such that $\sigma^k \to \sigma$ and, for all i and k, σ_i is a best response to every σ^k_{-i}

- Counterexample: (D,R) in the previous example

- Corollary: σ_i in a trembling-hand perfect equilibrium cannot be weakly dominated. No weakly dominated pure strategy can be played with positive probability

- Remark: the converse (any NE not involving weakly dominated strategies is trembling hand perfect) is true for two-player games but not for more than two

Existence

- Proposition: Every $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$ with finite strategy sets S_i has a trembling-hand perfect equilibrium
- Corollary: At least there is a NE in which no player plays any weakly dominated strategy with positive probability
- Counterexample: Bertrand-Nash equilibrium allowing for continuous set of prices

Proper Equilibrium

- Example:

	L	R
U	1, 1	0, 0
D	0, 0	0, 0

- NE: (U,L), (D,R). THP: (U,L) but not (D,R) (weakly dominated)
- But adding two weakly dominated strategies:

	L	M	R
U	1, 1	0, 0	-9, -9
M	0, 0	0, 0	-7, -7
D	-9,-9	-7,-7	-7, -7

- NE: (U,L), (M,M), (D,R). THP: (U,L), (M,M)

- (M,M) is THP:
  e.g. consider the totally mixed $(\varepsilon, 1-2\varepsilon, \varepsilon)$ for both
  for player 1 (or 2), deviating to U (or L): $(\varepsilon - 9\varepsilon) - (-7\varepsilon) = -\varepsilon < 0$

- Idea of proper equilibrium: more likely to tremble to better strategies:
  second-best actions assigned at most ε times the probability of third-best actions,
  fourth-best actions assigned at most ε times the probability of third-best actions,
  etc.

- (M,M) is not proper equilibrium:
  e.g. if player 2 puts weight ε on L and ε^2 on R
  for player 1, deviating to U: $(\varepsilon - 9\varepsilon^2) - (-7\varepsilon^2) > 0$ for ε small

- See Fudenberg and Tirole for a formal definition and properties

Towards Correlated Equilibria

- Example:

	a	b
A	9, 9	6, 10
B	10, 6	0, 0

- Pure and mixed strategy NE:

  (A, b) with payoffs $(6, 10)$

  (B, a) with payoffs $(10, 6)$

  $\left[\left(\frac{6}{7}, \frac{1}{7}\right), \left(\frac{6}{7}, \frac{1}{7}\right)\right]$ with expected payoffs $\approx (8.57, 8.57)$

New Potential Agreement (1)

- First potential agreement:
  appoint a third party to flip a coin and announce "H" or "T"
  play (A, b) if H and (B, a) if T
  expected payoffs $(8, 8)$

- Are these "mutual best responses"?
  If "H", player 1 knows that player 2 plays b, A is a best response
  If "T", player 1 knows that player 2 plays a, B is a best response
  same for player 2 (game is symmetric)

- Expected payoffs could dominate mixed NE (e.g. change 9s for 8s)

New Potential Agreement (2)

- Second potential agreement:
  ask third party to roll a dice (number n not observed by players) and
  announce player 1 whether n is in {1,2} or in {3,4,5,6} and
  announce player 2 whether n is in {1,2,3,4} or in {5,6}
  player 1 plays B if {1,2} and A if {3,4,5,6}
  player 2 plays a if {1,2,3,4} and b if {5,6}
  expected payoffs $(8.33, 8.33)$ (still lower than in mixed, but wait...)

- Are these mutual best responses? e.g. for player 1...
  if n is in {1,2}, she knows that 2 plays a and then B is a best-response
  if n is in {3,4,5,6}, she gives $\frac{1}{2}$ to both a and b, and then A is a best-response

Correlated Equilibrium: Definition

- Definition in a finite game $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$:
  σ^*, a probability distribution over $S_1 \times ... \times S_I$, is a correlated equilibrium iff for all i and for all s_i chosen with positive probability, s_i solves

$$\max_{s_i'} E_{s_{-i}} \left[u_i(s_i', s_{-i}) \mid s_i, \sigma^* \right]$$

- In the previous second potential agreement, the probability distribution was...

	a	b
A	1/3	1/3
B	1/3	0

Example (continued)

- More generally, consider the family:

	a	b
A	γ	$(1-\gamma)/2$
B	$(1-\gamma)/2$	0

- Is it a correlated equilibrium? e.g. for player 1...
  when told to play B, she knows that 2 plays a. B is a best response
  when told to play A, prob of a is $\frac{\gamma}{\gamma+(1-\gamma)/2} = \frac{2\gamma}{1+\gamma}$. A is a best response iff

$$9\left(\frac{2\gamma}{1+\gamma}\right)+6\left(\frac{1-\gamma}{1+\gamma}\right) \geq 10\left(\frac{2\gamma}{1+\gamma}\right)+0\left(\frac{1-\gamma}{1+\gamma}\right) \text{ or } \gamma \in [0,3/4]$$

- Remarks:
  $\gamma = 0$ corresponds to the previous first potential agreement
  $\gamma = 3/4$ has payoffs $(8.75, 8.75)$, dominating the mixed NE

Mixed NE and Correlated Equilibrium

- Interpretation of a mixed strategy equilibrium:
  players' randomisations are independent
  condition decisions on private and independent signals
  e.g. $(1/2, 1/2)$ in matching pennies: choose H if the first step of the day is with your right foot and T if it is with your left one

- Interpretation of correlated equilibrium:
  players' randomisations may be correlated
  decisions may also be conditioned on a public signal
  e.g. realisation of a flip of a coin in the previous first potential agreement