Lecture 5:
Hypothesis Testing

Quantitative Methods for
Regulation and Competition

Remember?

- We made inferences about an unknown parameter of the population based on the sample:
  - Point estimation: obtain an estimate of the parameter
  - Interval estimation: obtain a range within which we have a degree of certainty (confidence level) the parameter lies
- Example: mean of IQ levels of the student population:
  - Point estimate: 120
  - Interval estimate: (114,125)
- This was based on the Central limit theorem:
  - Sample mean of a population drawn from a distribution with mean μ and variance σ^2 has mean μ and variance σ^2/n.

Hypothesis Testing

- Now we want to test whether the mean is equal to a particular value
- Example: test whether the true IQ level is 110
- Procedure:
  - Define the hypothesis
  - Select a “good” estimator (e.g. the sample mean)
  - How different the estimated and the hypothesied values are?
    - If they are “close”, then we conclude that it is “plausible” that it is true!
    - If they are “far”, then we conclude that it is “surprising’ that it is true!

Today’s lecture

- Quick review of some of last week’s estimators
- Formalise the previous testing procedure:
  - Two-sided hypothesis testing
- Further concepts
- Extension:
  - One-sided hypothesis testing

- Reading: Chapter 8 in Ashenfelter et al.

Review: Interval Estimation for the Population Mean

- Suppose that n is large. Then…

$$\overline{X} \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right)$$

- And…

$$Z = \frac{\overline{X} - \mu_X}{\sqrt{\frac{\sigma_X^2}{n}}} = \frac{\overline{X} - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} \sim N(0,1)$$

- And therefore

$$P\left(\overline{X} - z_{1-\alpha/2}\frac{\sigma_X}{\sqrt{n}} \leq \mu_X \leq \overline{X} + z_{1-\alpha/2}\frac{\sigma_X}{\sqrt{n}}\right) = 1 - \alpha$$

Review: Interval Estimation for the Population Proportion

- We have a sample $X_1, \ldots X_n$ can take values 0 and 1
- The unknown proportion of 1's in the population is π
- Take the following estimator…

$$\hat{\pi} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Is it unbiased? What is the variance?
- If n is large, we have that..

$$\hat{\pi} \sim N\left(\pi, \frac{\pi(1-\pi)}{n}\right)$$

- And therefore…

$$P\left(\hat{\pi} - z_{1-\alpha/2}\frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}} \le \pi \le \hat{\pi} + z_{1-\alpha/2}\frac{\sqrt{\hat{\pi}(1-\hat{\pi})}}{\sqrt{n}}\right) = 1-\alpha$$

Review: Interval Estimation for the Population Variance

- Since…

$$\frac{(n-1)s_X^2}{\sigma_X^2} \sim \chi_{n-1}^2$$

- Then…

$$P\left(\chi_{\alpha/2,n-1}^2 \leq \frac{(n-1)s_X^2}{\sigma_X^2} \leq \chi_{1-\alpha/2,n-1}^2\right) = 1-\alpha$$

- Following the same procedure…

$$P\left(\frac{(n-1)s_X^2}{\chi_{1-\alpha/2,n-1}^2} \leq \sigma_X^2 \leq \frac{(n-1)s_X^2}{\chi_{\alpha/2,n-1}^2}\right) = 1-\alpha$$

Further: Interval Estimation of Difference Between Means

- Suppose that X and Y are two independent random variables, then…

$$\overline{X} \sim N\left(\mu_X, \frac{\sigma_X^2}{n_X}\right) \quad \text{and} \quad \overline{Y} \sim N\left(\mu_Y, \frac{\sigma_y^2}{n_Y}\right) \quad \text{and}$$

$$\overline{X} - \overline{Y} \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n_X} + \frac{\sigma_y^2}{n_Y}\right)$$

- And therefore

$$P\left(\overline{X} - \overline{Y} - \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}\, z_{1-\alpha/2} \leq \mu_X - \mu_y \leq \overline{X} - \overline{Y} + \sqrt{\frac{\sigma_X^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}\, z_{1-\alpha/2}\right) = 1 - \alpha$$

Hypothesis Testing

- Null hypothesis:
  - Denoted as H_0
  - Should be a specific value
  - Example: $H_0 : \mu=110$
- Alternative hypothesis:
  - Denoted as H_1
  - Can be two or one sided
  - Examples: $H_1 : \mu \neq 110$, $H_1 : \mu > 110$, $H_1 : \mu < 110$

Two-sided Alternative Test

- Suppose that we want to test:

  $H_0 : \mu = a$

  $H_1 : \mu \neq a$

- We know that, if n is large,…

$$P\left(-z_{1-\alpha/2} \leq \frac{\overline{X} - \mu_X}{\frac{\sigma_X}{\sqrt{n}}} \leq z_{1-\alpha/2} \right) = 1 - \alpha$$

And therefore if the hypothesis is true then…

$$P\left(-z_{1-\alpha/2} \leq \frac{\overline{X} - a}{\frac{\sigma}{\sqrt{n}}} \leq z_{1-\alpha/2} \right) = 1 - \alpha$$

Procedure: Two-sided Alternative Test

- Define the null hypothesis
  - Example: H_0 : μ=110
- Calculate the “test statistic”:
  - Example: $\frac{\overline{X}-a}{\frac{\sigma}{\sqrt{n}}}=\frac{120-110}{\frac{10}{\sqrt{16}}}=4$
- Decide on the confidence level and find the z’s:
  - Example: 95% and $z_{1-\alpha/2}$=1.96
- Check whether the “statistic” falls within the bounds defined by the z’s:
  - Example: No! 4>1.96. We should reject the null hypothesis

P-VALUES

- So far: reject the null hypothesis if the sample evidence is very unlikely were the null hypothesis true
- One could exactly compute the probability of observing that statistic were the null hypothesis true
- This value is called the p-value!

Type I and Type II Errors

- Type I error:
  - Reject the null hypothesis when, in fact, it was true
  - The probability that this happens is…
- Type II error:
  - Accept the null hypothesis when, in fact, it was false
  - The probability that this happens is denoted as ß
  - The *power* of the test is 1 - ß
- Of course both are linked!
  - If we try to reduce the type I error, we are increasing the type two error
  - In practice, we set $\alpha=0.05$ or $\alpha=0.10$

One-sided Alternative Test

- Suppose that we want to test:

  $H_0 : \mu = a$

  $H_1 : \mu > a$

- We know that, if n is large: $$P\left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq z_{1-\alpha}\right) = 1-\alpha$$

- And therefore if the hypothesis is true: $$P\left(\frac{\overline{X}-a}{\frac{\sigma}{\sqrt{n}}} \leq z_{1-\alpha}\right) = 1-\alpha$$

- Reject the null whenever: $$\frac{\overline{X}-a}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha}$$