Chapter 3: Production Theory

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3.- Producer Theory

3.1.- Production functions: physical relation between inputs and outputs

3.2.- Cost minimisation: prices and input choice

3.3.- Profit maximisation and supply function

3.1 Production Functions

- Marginal and average physical productivity
- Isoquants and marginal rate of technical substitution
- Returns to scale
- Elasticity of substitution
- Examples of production functions

Input and Output

- Definition: A firm's *production function* for a given good q is given by

$$q = f(L, K, M...)$$

- Shows maximum amount of output that can be produced in a given period for alternative combinations of labour (L), capital (K), raw material (M),...

- Example:

$$q = 600K^2L^2 - K^3L^3$$

Marginal and Average Physical Productivity

- *Marginal physical product of an input*: (=additional output obtained)

$$MP_L \equiv \frac{\partial q}{\partial L} = \frac{\partial f(L, K, M...)}{\partial L},\ MP_K \equiv \frac{\partial q}{\partial K} = \frac{\partial f(L, K, M...)}{\partial K}...$$

- Can it be negative? Sometimes assumed to be diminishing (interpretation?):

$$\frac{\partial MP_K}{\partial K} = \frac{\partial^2 q}{\partial^2 K} = \frac{\partial^2 f(L, K, M...)}{\partial^2 K} < 0,\ ...$$

- *Average physical product of an input*:

$$AP_K \equiv \frac{q}{K} = \frac{f(L, K, M...)}{K},\ ...$$

- Average productivity for labour often called "productivity" (measure of efficiency)

Example

- For the production function

$$q = 600K^2L^2 - K^3L^3$$

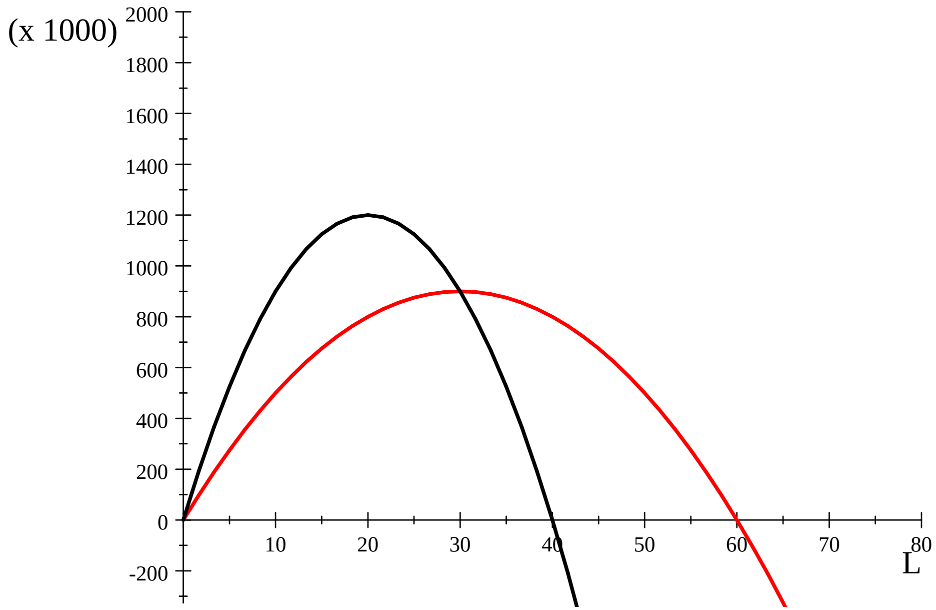
- The marginal physical products of capital and labour are

$$MP_K = 1200KL^2 - 3K^2L^3 \text{ and } MP_L = 1200K^2L - 3K^3L^2$$

- Their average physical products are

$$AP_K = 600KL^2 - K^2L^3 \text{ and } AP_L = 600K^2L - K^3L^2$$

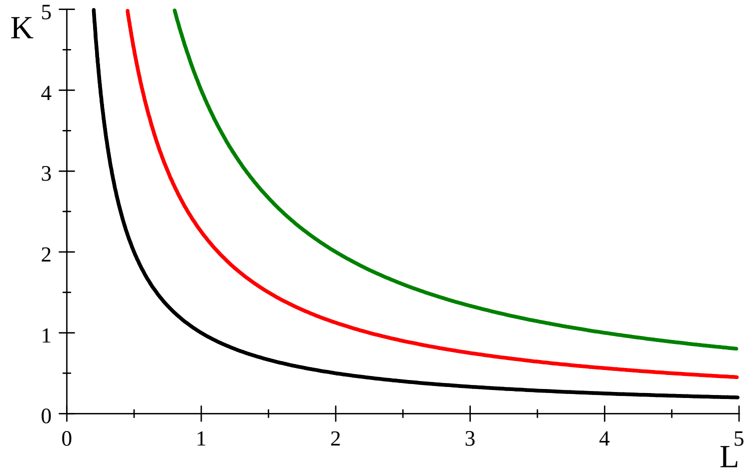
Graphically



- MP_L (black) and AP_L (red) for $K = 10$
- Maximum production? Maximum “productivity”?

Isoquant Maps

- Isoquant: combinations of inputs giving the same output q_o
- Example: $q = f(L, K) = 10L^{1/2}K^{1/2}$ and $q_o = 10, 15, 20$ (black, red, green)



- Similar to indifference curves but now iso(-quantity) levels are measurable

Marginal Rate of Technical Substitution (RTS)

- RTS: rate at which labour can be substituted for capital holding output constant

$$RTS_{L,K}(L,K) \equiv -\left.\frac{\partial K}{\partial L}\right|_{q=q_o} = \frac{MP_L}{MP_K}$$

- Example: $q = f(L,K) = 10L^{1/2}K^{1/2}$

$$RTS_{L,K}(L,K) = \frac{5L^{-1/2}K^{1/2}}{5L^{1/2}K^{-1/2}} = \frac{K}{L}$$

(a) $RTS(1,4) = 4$, (b) $RTS(1,1) = 1$ and (c) $RTS(4,1) = 1/4$

- Interpretation (a): for extra unit of labour, we can dispense of 4 of capital

RTS and Marginal productivity

- Diminishing marginal productivities imply diminishing RTS? Not necessarily!
- Take $K(L)$ and denote f_i and f_{ij} first and second derivatives wrt i and j :

$$\frac{\partial RTS_{L,K}(L,K)}{\partial L} = \frac{f_K\left(f_{LL} + f_{LK}\frac{\partial K}{\partial L}\right) - f_L\left(f_{KL} + f_{KK}\frac{\partial K}{\partial L}\right)}{(f_K)^2}$$

and since $\frac{\partial K}{\partial L} = -\frac{f_L}{f_K}$:

$$\frac{\partial RTS_{L,K}(L,K)}{\partial L} = \frac{(f_K)^2 f_{LL} - 2f_K f_L f_{LK} + (f_L)^2 f_{KK}}{(f_K)^3}$$

- Since $f_K > 0$ and f_{KK}, $f_{LL} < 0$, RTS diminishing if $f_{LK} > 0$. Interpretation?

Returns to Scale

- How does output change to proportional increase in *all* inputs?
  (i) more than proportionally (greater division of labour and specialisation) or
  (ii) less than proportionally (managerial overseeing more difficult)?

- For $m > 1$, the *returns to scale* of production are
  constant if $f(mL, mK) = mf(L, K) = mq$
  increasing if $f(mL, mK) > mf(L, K) = mq$
  decreasing if $f(mL, mK) < mf(L, K) = mq$

- Real production functions often first increasing and then decreasing

- Why returns to scale are so important?

Cobb-Douglas Example

- Suppose output could be produced by capital and labour according to

$$q = f(L, K) = AL^{\beta}K^{\alpha}$$

where $A > 0$ and α,$\beta > 0$ are elasticities of output wrt capital and labour input

- Given that

$$f(mL, mK) = A\,(mL)^{\beta}\,(mK)^{\alpha} = Am^{\alpha+\beta}L^{\beta}K^{\alpha} = m^{\alpha+\beta}f(L, K)$$

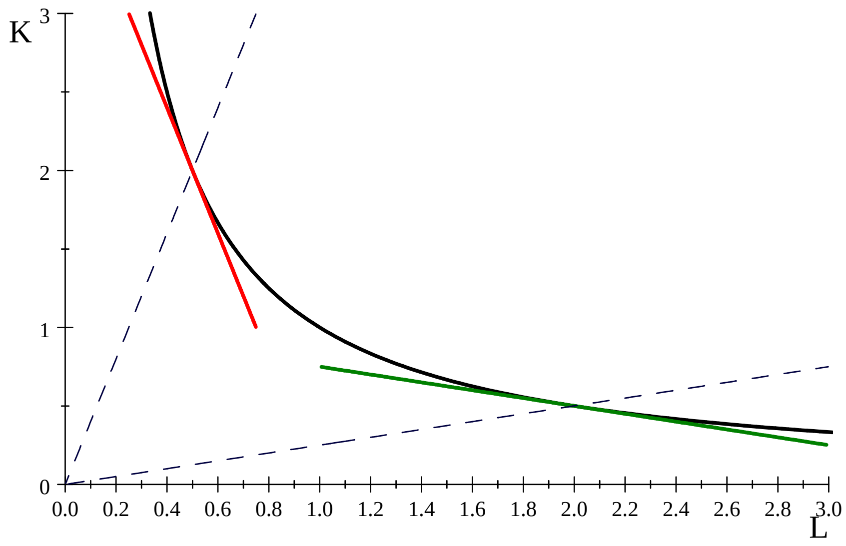
constant, increasing and decreasing returns to scale iff $\alpha + \beta = 1$, > 1, < 1

- Assuming Cobb-Douglas, A, α and β can be estimated econometrically:

$$\ln q = \ln A + \beta \ln L + \alpha \ln K + \varepsilon$$

Ease of Substitution along Isoquant

- Along isoquant RTS decreases but, by how much?
  Measure relative changes in K/L with respect to changes in RTS



Elasticity of Substitution

- Definition:

$$\sigma \equiv \frac{\partial (K/L)}{\partial RTS}\frac{RTS}{(K/L)} = \frac{\partial \ln K/L}{\partial \ln RTS}\left(= \frac{\text{percent } \Delta (K/L)}{\text{percent } \Delta RTS}\right)$$

- Typically $\sigma > 0$ (might vary along and across isoquants)

- Example: Cobb-Douglas $q = f(L, K) = AL^{\beta}K^{\alpha}$

$$RTS = \frac{\beta K}{\alpha L}$$

and therefore

$$\ln RTS = \ln \frac{\beta}{\alpha} + \ln \frac{K}{L} \text{ and } \sigma = 1$$

Examples of production functions (1)

- Linear production function

$$q = f(L, K) = \alpha K + \beta L$$

for α,$\beta > 0$

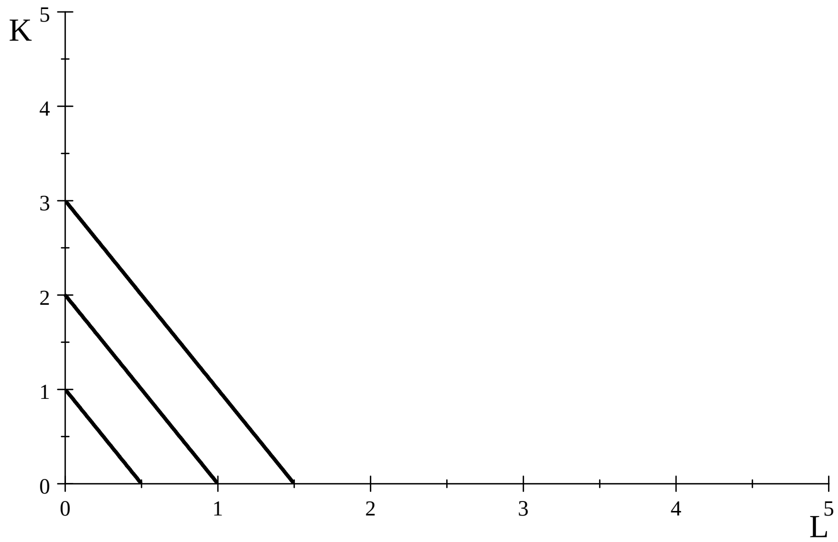
- Has constant returns to scale because for $m > 0$

$$f(mL, mK) = \beta mL + \alpha mK = mf(L, K)$$

- $RTS =$?, $\sigma =$?

- Reasonable? Examples?

Linear production function with $\alpha = 2$ and $\beta = 1$



Examples of production functions (2)

- Fixed proportions

$$f(L, K) = \min\{\beta L, \alpha K\}$$

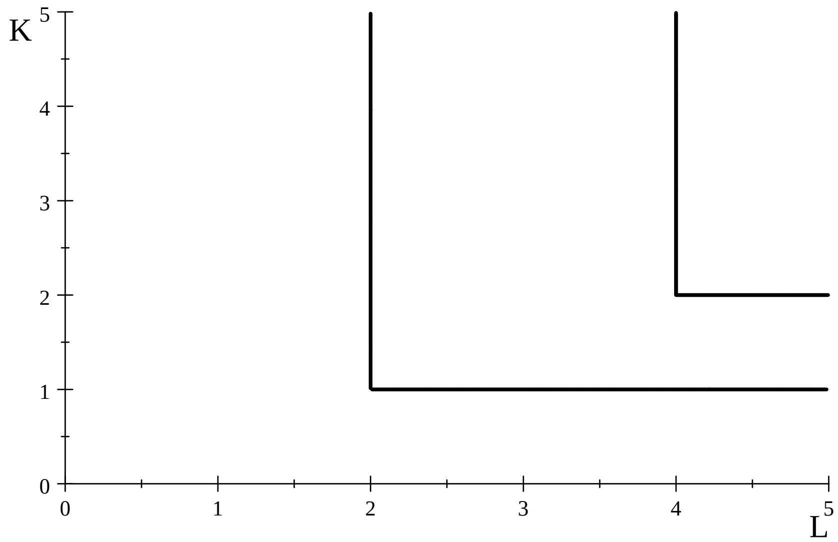
  for α,$\beta > 0$.

- A firm with this production will always operate at the corner

- Returns to scale?

- Examples?

Fixed proportions with $\alpha = 2$ and $\beta = 1$



Examples of production functions (3)

- Constant elasticity of substitution (CES) production function

$$q = f(L, K) = [K^\rho + L^\rho]^{\epsilon/\rho}$$

where $\rho \leq 1$, $\rho \neq 0$, $\epsilon > 0$

- Returns to scale? Marginal rate technical substitution

$$RTS = \frac{(\epsilon/\rho)\, q^{1-\rho/\epsilon} \rho L^{\rho-1}}{(\epsilon/\rho)\, q^{1-\rho/\epsilon} \rho K^{\rho-1}} = \frac{L^{\rho-1}}{K^{\rho-1}} = \left(\frac{K}{L}\right)^{1-\rho}$$

and therefore

$$\ln RTS = (1-\rho) \ln \frac{K}{L} \text{ and } \sigma = \frac{\partial \ln K/L}{\partial \ln RTS} = \frac{1}{1-\rho}$$

- Particular cases: linear, fixed proportions and Cobb-Douglas ($\rho = 1$, -∞, 0 resp.) (proof requires limit arguments, see textbook)

Production vs utility function?

- Output from inputs vs utility level from purchases
- Derived from technologies vs derived from preferences
- Cardinal (unique and specific amount) vs ordinal
- Marginal Product vs marginal utility
- Isoquant vs indifference curve
- Marginal rate of technical substitution vs marginal rate of substitution

3.2.- Cost Minimisation

- Definition of costs
- Cost minimisation choices
- Input demand and total cost function
- Average and marginal costs functions
- Shifts in cost curves
- Short and long run cost functions

Input costs, but which costs?

- “Accounting” costs: out-of-pocket historical costs appropriately depreciated
- “Economic” (or opportunity) costs: payment necessary to keep resources in place or, equivalently, the remuneration received in the next best alternative
- Example: bought steel for £1m but price has gone up and is now worth £1.2m
  Economic and accounting costs?
- We’ll always talk about economic costs
- Most of the time here, for simplicity, homogenous labour and capital costs
  Denote w labour cost (labour-hour) and r capital cost (price of machine-hour)

Costs minimisation

- Firm selects K and L to minimise costs of producing a given q_o

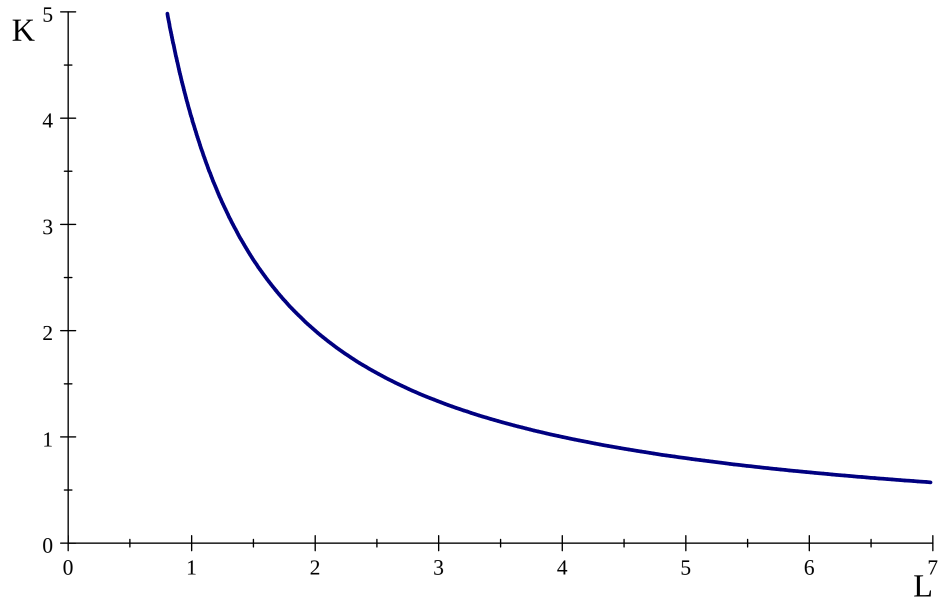
$$TC \equiv wL + rK$$

- First step towards maximisation of profits,

$$\pi \equiv TR - TC$$

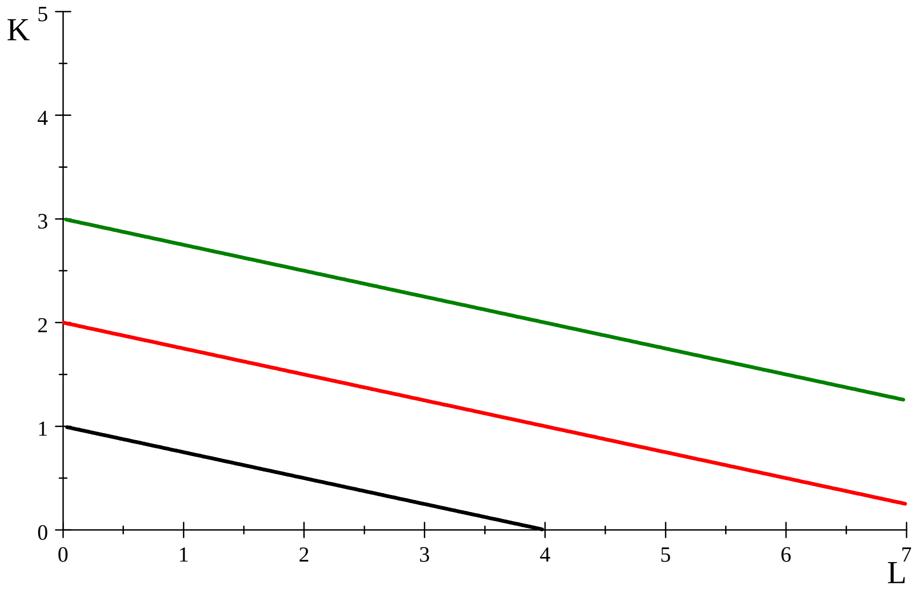
- But, it can accommodate more general objectives (e.g. regulated transmission firms are required to transport a certain quantity)

Required production level



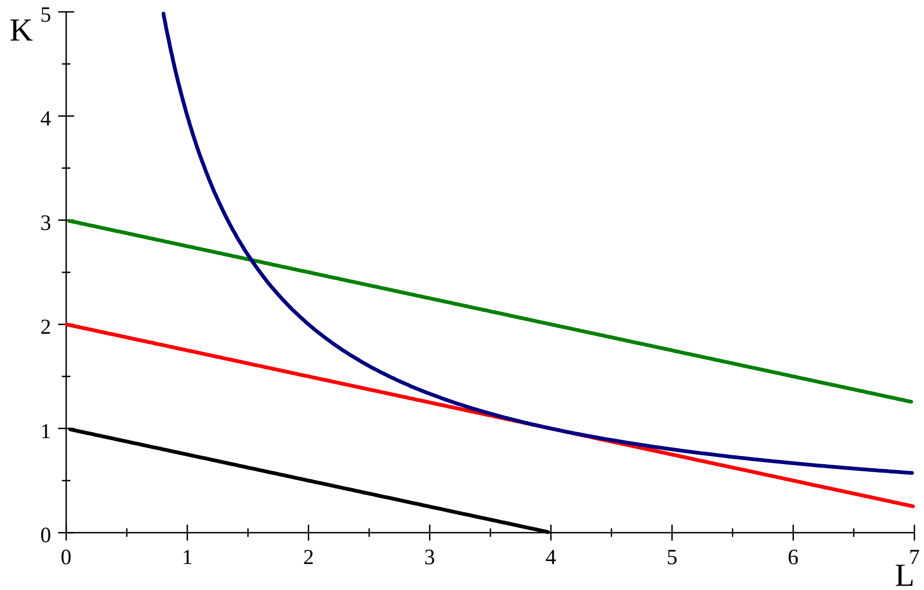
- Example: $q = f(L, K) = 10K^{1/2}L^{1/2}$ and $q_o = 20$ units are required

Isocost lines



- Slope $-w/r$, vertical intercept TC/r and horizontal TC/w
- Graph: $w = 0.25$ and $r = 1$, $TC = 3$ (green), 2 (red) and 1 (black)

Which input combination?



- Optimal combination is such that its isoquant is tangent to its isocost line

Mathematical Problem

- Find K^*, L^* that solve

$$Min\ TC \equiv wL + rK$$
$$subject \text{ to } q = f(L, K) = q_o$$

- This can be found by solving

$$Min_{L,K}\ \mathcal{L} = \ wL + rK + \lambda\left[q_o - f(L, K)\right]$$

or

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \frac{\partial f(L^*, K^*)}{\partial L} = 0 \text{ and } \frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{\partial f(L^*, K^*)}{\partial K} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = q_o - f(L^*, K^*) = 0$$

Input demand and total cost function

- Solution satisfies

$$RTS(L^*, K^*) = \frac{\frac{\partial f(L^*,K^*)}{\partial L}}{\frac{\partial f(L^*,K^*)}{\partial K}} = \frac{w}{r} \text{ and } q_o = f(K^*, L^*)$$

- One obtains the *input demands* and the *total cost function*:

$$L^*(w, r, q_o) \text{ and } K^*(w, r, q_o) \text{ and } TC^*(w, r, q_o)$$

- Similar than consumer's expenditure minimisation problem

- Dual problem: output maximisation for a given level of total costs

Hamburger production

- Hourly production (q) depends on number of grills (K) and worker-hours (L)

$$q = 10K^{1/2}L^{1/2}$$

  and management wants to produce q_o hamburgers per hour

- Optimal combination satisfies

$$RTS(L^*, K^*) = \frac{K^*}{L^*} = \frac{w}{r} \text{ and } q_o = 10\left(K^*\right)^{1/2}\left(L^*\right)^{1/2}$$

- Input demands and total cost function:

$$L^*(w, r, q_o) = \frac{q_o}{10}\left(\frac{r}{w}\right)^{1/2} \text{ and } K^*(w, r, q_o) = \frac{q_o}{10}\left(\frac{w}{r}\right)^{1/2}$$

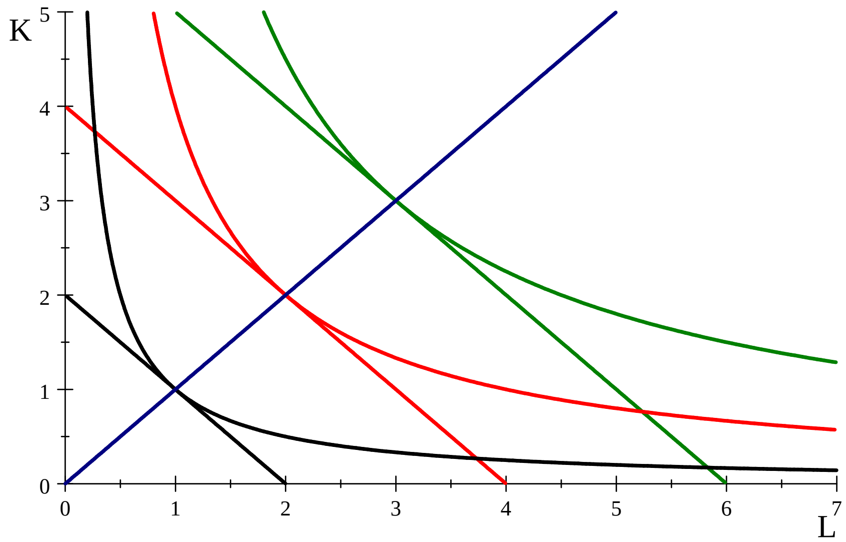
- If $q_o = 20$, $w = 0.25$, $r = 1$, then $L^* = 4$ and $K^* = 1$

Price elasticity for input demand

Input	Capital	Production	Non-production	Electricity
Industry		labour	labour	
Textiles	-0.41	-0.5	-1.04	-0.11
Paper	-0.29	-0.62	-0.97	-0.16
Chemicals	-0.12	-0.75	-0.69	-0.25
Metals	-0.91	-0.41	-0.44	-0.69

- Barnett, Reutter and Thompson, "Electricity Substitution: Some Local Industrial Evidence" Energy Economics 1998

Firm's expansion path



- Higher quantity required (10, 20, 30) implies higher costs (black, red, green)
- Normal (inferior) input if input demand increases (decreases) with output

Average and marginal cost function

- Total cost function (change q for q_o and take * out) is then

$$TC^*(w,r,q_o) \equiv TC(w,r,q) = \frac{q}{5}(wr)^{1/2}$$

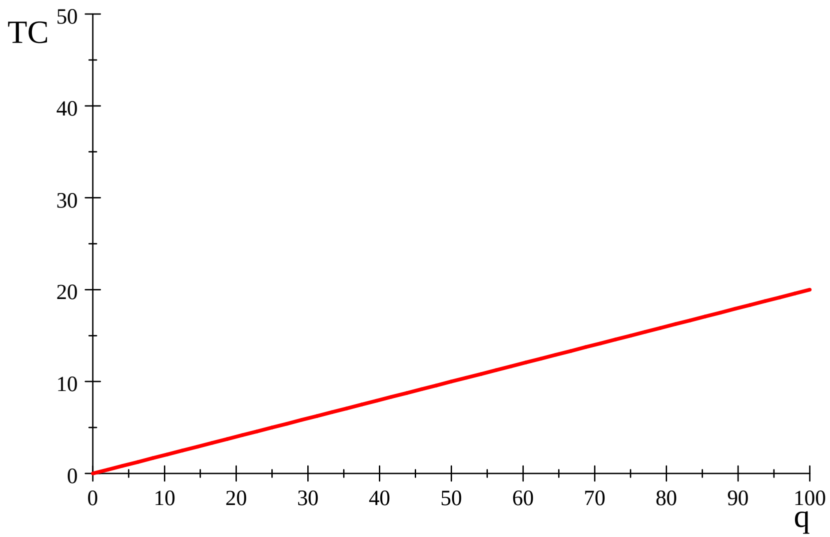
and therefore average cost function

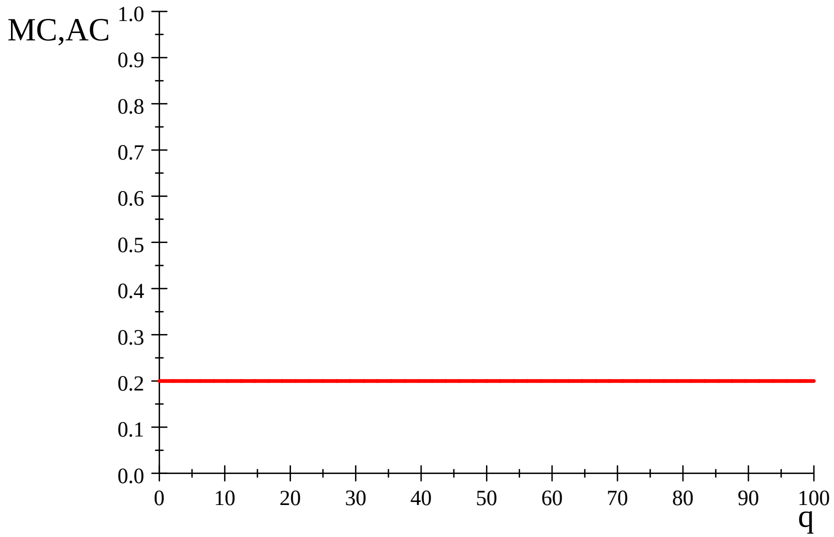
$$AC(w,r,q) = \frac{TC(w,r,q)}{q} = \frac{1}{5}(wr)^{1/2}$$

and the marginal cost function

$$MC(w,r,q) = \frac{\partial TC(w,r,q)}{\partial q} = \frac{1}{5}(wr)^{1/2}$$

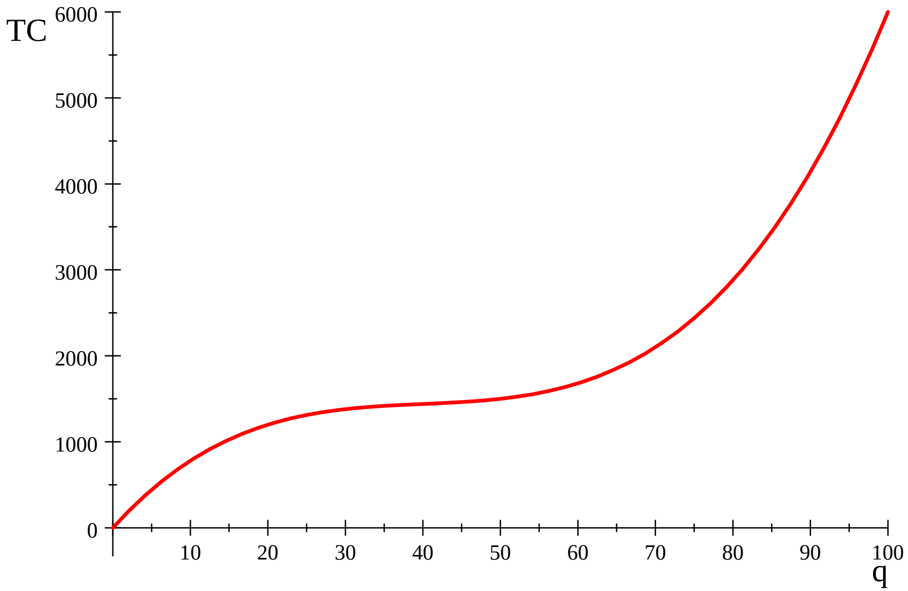
Hamburgers example





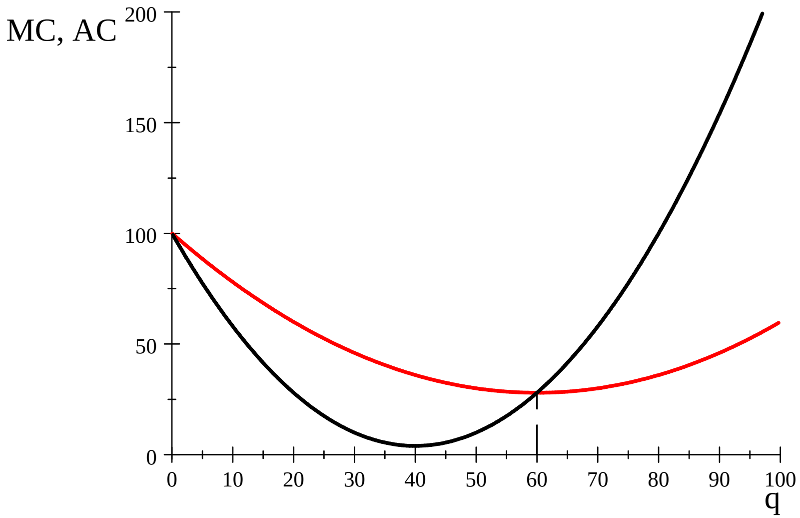
- Remember $q_o = 20$, $w = 0.25$, $r = 1$

Another total cost curve example



- Total (cubic) cost curve (for given input prices): $TC(q) = 100q - 2.4q^2 + 0.02q^3$
  Can you hint the "returns to scale" here?

Example: MC and AV



- $MC(q) = 100 - 4.8q + 0.06q^2$ (black) $AC(q) = 100 - 2.4q + 0.02q^2$ (red)
- MC and AV coincide at the origin, why? AC is then higher than MC, why? If $AC > MC$ then $AC \downarrow$, why? Intersection at minimum of AC (60), why?

Economies of scale

- Cost exhibits economies of scale if AC $\downarrow$: (=increasing returns to scale)
- Cost exhibits diseconomies of scale if AC $\uparrow$: (=decreasing returns to scale)
- Minimum of AC is called minimum efficiency scale (MES).
- MES for selected US food and beverage industries

Industry	MES as % of market output
Beet sugar (processed)	1.87
Cane sugar (processed)	12.01
Flour	0.68
Breakfast cereal	9.47
Baby food	2.59

Cost curve shifts

- If all prices are multiplied by a factor t input choices do not change (why?):

$$L^*(tw, tr, q_o) = L^*(w, r, q_o) \text{ and } K^*(tw, tr, q_o) = K^*(w, r, q_o)$$

- Total costs are multiplied by t

$$TC^*(tw, tr, q_o) = tTC^*(w, r, q_o)$$

- Possible to derive input substitution effects if only one price changes
- Typically, an increase in an input's price decreases its use and increases the other input's use (sufficient condition for this?)

Long and short run costs

- Not precise time distinction but related to more and less flexibility
- Here one input (e.g. capital) can be fixed (e.g. at K_1 units) in short-run
- Short-run total costs as a function of input:

$$STC \equiv rK_1 + wL,$$

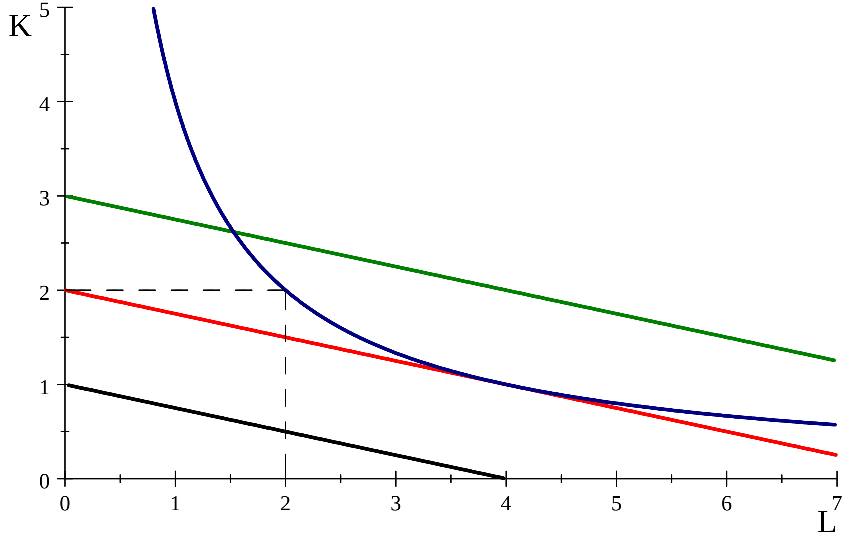
comprised of short-run fixed and variable costs

$$STC \equiv SFC + SVC$$

- Fixed costs are paid no matter how much is produced
- What will be the short-run input choices?

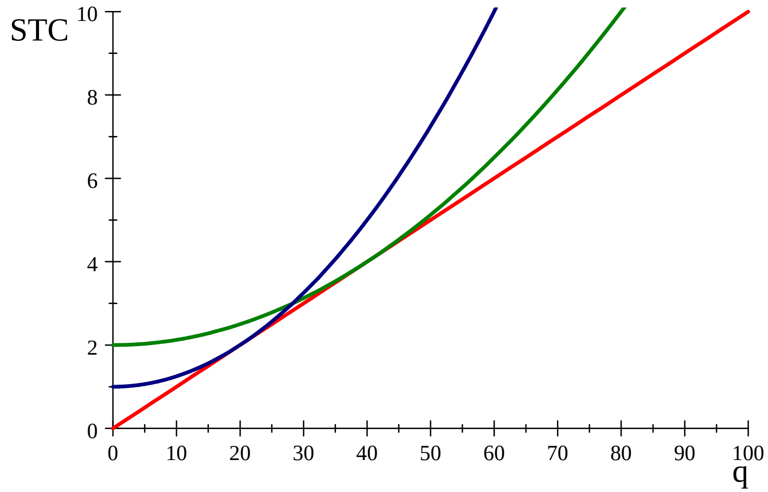
Hamburgers again

- Cobb-Douglas, $f(L,K) = 10K^{1/2}L^{1/2}$, $w = 0.25$, $r = 1$, $q_o = 20$, $K_1 = 2$



- Input combination? Higher or lower total costs than in the long-run?

Short run cost function

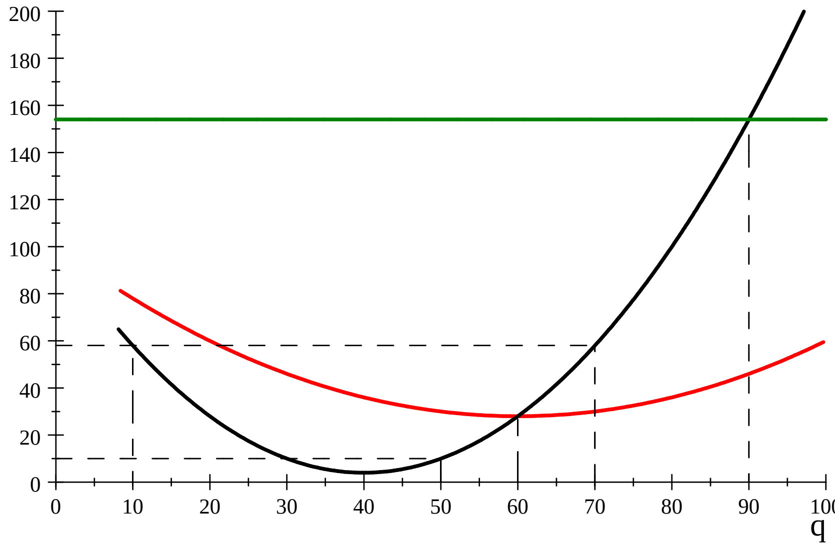


- If $10K_1^{1/2}L^{1/2} = q_o$ then $L^* = q_o^2/(100K_1)$ and $STC = rK_1 + wq_o^2/(100K_1)$
- STC for $K_1 = 2$ (green) & $K_1 = 1$ (blue). Why intersect once with TC (red)?

3.3.- Profit maximisation and supply

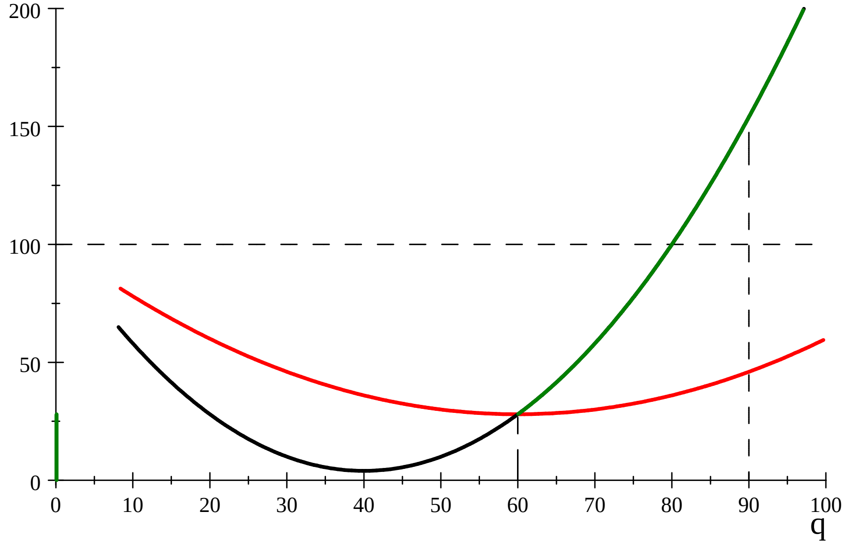
- Firm transforms inputs into outputs
- Assume profit maximisation objective. Why?
- First maximising profits if $TC(q_o, r, w)$ known and output prices given
- Maximising profits directly choosing inputs

(Long-run) profit maximisation



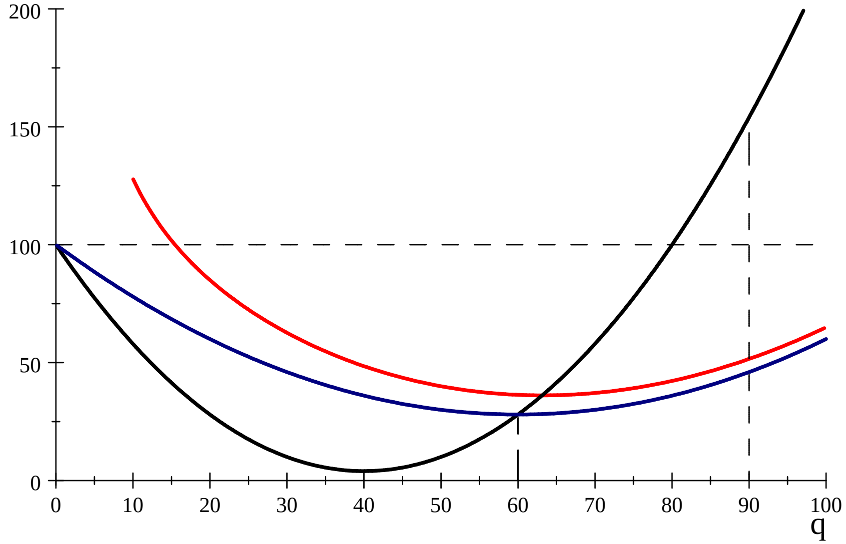
- $P = 154$ (green), AC (red), MC (black). Optimal quantity? Why?
- And if $P = 58$? $P = 10$? Representation of profits?

(Long-run) supply function (green)



- How would you construct supply function if an input (e.g. capital) is fixed?
- What if fixed costs can be avoided if nothing is produced (non-sunk)?
  Fixed costs can be either sunk or non-sunk. Examples?

Short run supply function



- Take new example. Now SAC (red), SMC (black) and $SAVC$ (blue)
- Supply function depending on whether fixed costs are sunk/not sunk?

Hamburgers again

- Remember that $STC(r, w, q_o) = rK_1 + wq_o^2/(100K_1)$ (fixed costs sunk)
- If $w = 0.25$ and $r = 1$, $K_1 = 2$, then

$$STC = 2 + 0.25q_o^2/200$$

and therefore

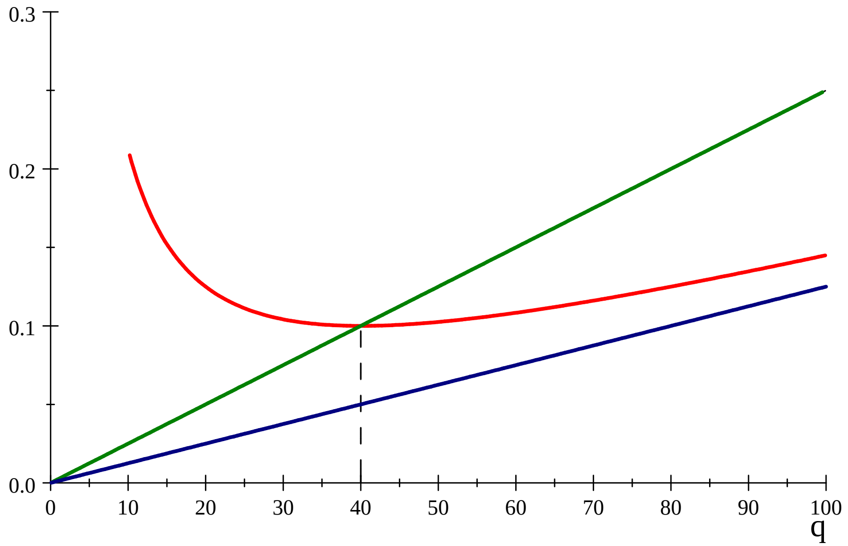
$$SMC = 0.5q_o/200 \text{ and } SAC = 2/q_o + 0.25q_o/200$$

and

$$SVC = 0.25q_o^2/200 \text{ and } SAVC = 0.25q_o/200$$

- Given that $SMC > SAVC$ for any q_o, supply... ?

Short-run hamburger supply



- SAC (red), SMC (black), $SAVC$ (blue). Might the firm lose money?

Profit maximisation and input demand

- Now maximise profits by choosing inputs (L^*, K^*) directly

$$\pi \equiv TR - TC \equiv Pf(L, K) - (wL + rK)$$

where P price of the output

- Unconstrained maximisation! Find L^* and K^* such that

$$\frac{\partial \pi}{\partial L} = P\frac{\partial f(L^*, K^*)}{\partial L} - w = 0 \text{ and } \frac{\partial \pi}{\partial K} = P\frac{\partial f(L^*, K^*)}{\partial K} - r = 0$$

- Intuition? Costs are minimised. Indeed, again

$$RTS(L^*, K^*) = \frac{w}{r}$$

- Substituting $L^*(w, r, P)$ and $K^*(w, r, P)$ into f

$$q^* = f(L^*, K^*) = q^*(w, r, P)$$