Chapter 3: Best Response and Nash

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Best Response Functions

- So far, we have examined all combination profiles to find whether they were NE
- If there are many possible actions and many players, this might be difficult
- Alternative: find best action for a player to any given list of others' actions
  BoS game: for 1, Bach is a best response if 2 chooses Bach and Stravinsky is a best response if 2 chooses Stravinsky
- Best response to a given action of the others may not be unique. Example: for 1, T and B are best responses if 2 chooses L

	L	M	R
T	1,1	1,0	0,1
B	1,0	0,1	1,0

- Formally, denote best response for i to the actions of the others a_{-i} as $B_i(a_{-i})$
  e.g. $B_1(Bach) = Bach$ and $B_1(Stravinsky) = Stravinsky$ in Bos
  e.g. $B_1(L) = \{T, B\}$, $B_1(M) = \{T\}, B_1(R) = \{B\}$ in previous game

Best Response Functions and Nash

- Remember that in a NE, no player can do better than playing her NE strategy if the others play their NE strategies
- Hence, we can redefine the concept of NE in terms of best response functions
- Definition: $(s_1, ..., s_I)$ is a NE if and only if every player's action is a best response to the other players' actions, i.e. s_i is in $B_i(s_{-i})$ for every i
- Method to find NE: (a) Find best response functions and (b) Find strategy profiles that are mutually best responses:

	L	C	R
T	1,2	2,1	1,0
M	2,1	0,1	0,0
B	0,1	0,0	1,2

- If best-responses have 1 element: $(s_1, ., s_I)$ is a NE iff $B_i(s_{-i}) = s_i$ for every i
- This is a system of I equations with I unknowns (s_i's)

Example: A Synergistic Relationship

- Two individuals in synergistic relationship: if both devote effort, both better off
- Suppose that the relationship value for i ($i = 1, 2$) is given by $a_i(c + a_j - a_i)$, where a_i and a_j are own and other's efforts, resp.
- Players: $1, 2$. Strategies: $a_i > 0$ and payoffs: $a_i(c + a_j - a_i)$ (for $i = 1, 2$)
- Best response: suppose player j plays a_j, what is my best response?
- Compute (partial) derivative and equate to 0 (check second order condition)

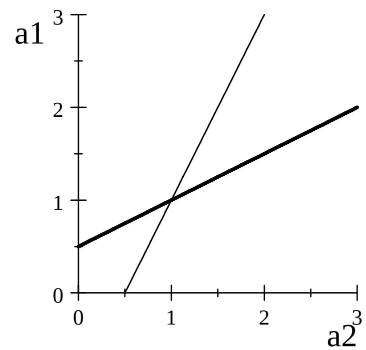
$$c + a_j - 2a_i^* = 0; \quad B_i(a_j) \equiv a_i^* = \frac{c + a_j}{2}$$

- Nash equilibrium: intersection of best reply functions, i.e. solve for

$$a_1^* = \frac{c + a_2^*}{2} \text{ and } a_2^* = \frac{c + a_1^*}{2}; \text{ and therefore } a_1^* = a_2^* = c$$

Example: A Synergistic Relationship (2)

- Suppose that $c = 1$. Best responses are $B_1(a_2) = \frac{1+a_2}{2}$ and $B_2(a_1) = \frac{1+a_1}{2}$. NE is: $a_1^* = a_2^* = 1$. Representation:



Best response functions: $B_1(a_2)$ (thick line, from horizontal to vertical axis) and $B_2(a_1)$ (thin line, from vertical to horizontal axis)
Nash equilibrium: $(1, 1)$ (the intersection of the best response functions)

Nash Equilibrium and Dominance

- A strictly dominated action is not a best response to any action
- Therefore, a strictly dominated action is not used in any Nash equilibrium
- One can eliminate strictly dominated actions when looking for NE
e.g. in PD, DC is never part of a NE: only possible NE is (C,C) (indeed, it is!)
- Can an action of a NE be weakly dominated? Yes!. Examples:

	B	C
B	1,1	0,0
C	0,0	0,0

	B	C
B	1,1	2,0
C	0,2	2,2

- C is weakly dominated in both games but (C,C) is a NE in both games
- (B,B) is a NE in both games. In the left game, this NE is better for both than the previous NE whereas in the right game, it is worse

Matching Pennies (MP)

- Some games do not have any NE. Example:

1\2	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- But here, players may want to introduce random behaviour
- Other examples include government auditing taxpayers
- Can we find an equilibrium when randomisation is allowed?

Mixed Strategies

- Notation: $s_i \in S_i$ (deterministic) *pure* strategy and S_i set of pure strategies
- Definition: A mixed strategy for player i, σ_i, assigns to each pure strategy s_i a probability $\sigma_i(s_i)$ that it will be played (where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$)
- Examples: in MP, (a) σ_i' such that $\sigma_i'(H) = \frac{1}{2}$, $\sigma_i'(T) = \frac{1}{2}$ or $\sigma_i' = (\frac{1}{2}, \frac{1}{2})$ (b) $\sigma_i'' = Head$, i.e. $\sigma_i''(H) = 1$, $\sigma_i''(T) = 0$ or $\sigma_i'' = (1, 0)$
- Definitions: set of mixed strategies: $\Delta(S_i)$, strategy profile: $\sigma = (\sigma_1, ..., \sigma_I)$.
- $u_i(\sigma)$, i's expected payoff to the mixed strategy profile σ, $u_i(\sigma) \equiv E_\sigma[u_i(s)]$
- Examples:
$u_1(\sigma_1', \sigma_2') = \frac{1}{2}\frac{1}{2}u_1(H, H) + \frac{1}{2}\frac{1}{2}u_1(H, T) + \frac{1}{2}\frac{1}{2}u_1(T, H) + \frac{1}{2}\frac{1}{2}u_1(T, T) = 0$
$u_1(\sigma_1', \sigma_2'') = \frac{1}{2}1u_1(H, H) + \frac{1}{2}0u_1(H, T) + \frac{1}{2}1u_1(T, H) + \frac{1}{2}0u_1(T, T) = 0$
$u_1(\sigma_1'', \sigma_2'') = 1{*}1u_1(H, H){+}1{*}0u_1(H, T){+}0{*}1u_1(T, H){+}){*}0u_1(T, T) = 1$

Mixed Strategy Nash Equilibrium

- We can redefine again a game (in normal form) as $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$

- Definition: A strategy profile $(\sigma_1, \sigma_2, ..., \sigma_I)$ constitutes a mixed strategy Nash equilibrium if for every $i = 1, ..., I$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for every } \sigma_i' \in \Delta(S_i)$$

- Example: (σ_1', σ_2') where $\sigma_i' = (\frac{1}{2}, \frac{1}{2})$ is a MSNE in MP

- But, is there any other?

- Alternative definition of NE: $(\sigma_1, \sigma_2, ..., \sigma_I)$ is a MSNE iff strategy σ_i is a best response to σ_{-i} for all i

- Example: σ_1' is a best response to σ_2' and vice versa in MP

Pure and Mixed Strategy NE

- Example: another coordination game ("Meeting in New York"):

$1\backslash 2$	ES	GC
ES	100,100	0,0
GC	0,0	1000,1000

- (ES, ES) and (GC, GC) are two (pure strategy) Nash equilibrium

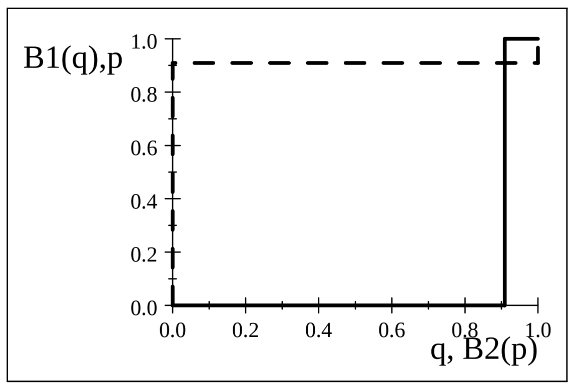
- Is there any mixed strategy Nash equilibrium as well? How do we find it?

A Method to Find Pure and Mixed Strategy NE

1. Find best-response functions. For each $[(p, 1-p), (q, 1-q)]$

$$B_1(q) = \begin{cases} 0 \text{ if } q \in [0, \frac{10}{11}) \\ [0,1] \text{ if } q = \frac{10}{11} \\ 1 \text{ if } q \in (\frac{10}{11}, 1] \end{cases} \quad \text{and } B_2(p) = \begin{cases} 0 \text{ if } p \in [0, \frac{10}{11}) \\ [0,1] \text{ if } p = \frac{10}{11} \\ 1 \text{ if } p \in (\frac{10}{11}, 1] \end{cases}$$

2. For two player-two action games, represent them and find intersections



A Method to Find Pure and Mixed Strategy NE (2)

- Or, assume that $[(p^*, 1-p^*), (q^*, 1-q^*)]$ is a MSNE and look for conditions:

  If $p^* = 0$ then $q^* = 0$. If $q^* = 0$ then $p^* = 0$. Hence $[(0,1),(0,1)]$ is a NE

  Similarly $[(1,0),(1,0)]$ is a NE.

  If $0 < p^* < 1$ then $q^* = \frac{10}{11}$ and then since $0 < q^* < 1$, $p^* = \frac{10}{11}$.

  Hence, $[(\frac{10}{11}, \frac{1}{11}), (\frac{10}{11}, \frac{1}{11})]$ is a NE

- Exercise: show that (σ_1', σ_2') is the unique MSNE in MP

- Notice that in the strictly MSNE, each player is indifferent among all the pure strategies played with positive probability

- This is a general property, see Osborne (2004)

- Proposition: Every static game in which players have finitely many actions has a MSNE