Chapters 5: Static Games

Financial Microeconomics

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Roadmap: part 2 of the course

- (5) Static Games
- (6) Dynamic Games

Introduction: Monopoly Pricing

- In the competitive model, many sellers that act as price takers
- In most markets, however, companies can profitably increase prices
- Suppose, at the other extreme, only one seller, a monopolist
- Suppose inverse demand function $D(q)$ is decreasing when positive:

$$D(q) = \begin{cases} 13 - q \text{ if } q \leq 13 \\ 0 \quad \text{if} \quad q > 13 \end{cases}$$

- Suppose that the costs of producing q units, $C(q)$, are increasing
- For example, assume that they are linear, $C(q) = q$

- Firm selects quantity that maximises profits, revenues minus costs

$$Max_q D(q)q - C(q)$$
$$Max_q (13 - q)q - q$$

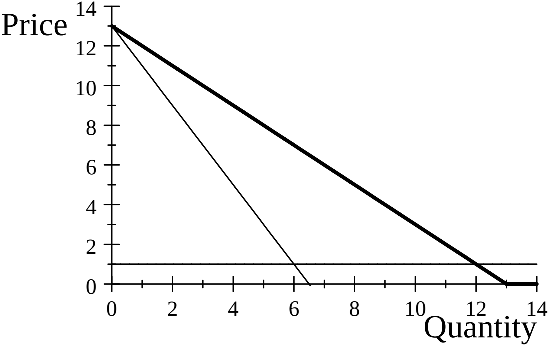
- The firm would select, q^m such that

$$\begin{aligned} -q^m + (13 - q^m) - 1 &= 0 \\ 13 - 2q^m - 1 &= 0 \\ q^m &= 6 \end{aligned}$$

- And therefore

$$D(q^m) = D(6) = 13 - 6 = 7$$

- Representation



Demand, marginal revenue and marginal cost

- $q^m = 6$, lower than the quantity s.t. demand=marginal cost, 12
- q^m s.t. marginal revenue=marginal cost, $13 - 2q^m = 1$
- Then, price, 7, higher than the competitive price, 1
- Monopoly selecting price instead of quantity would give the same

"Game Theory" Motivation: Multiple Sellers

- What happens when there are more than one firm (but not many)?
- The demand (and therefore profits) of each firm also depends on the quantity placed (or the price set) by the others
- More generally, decision-maker well-being also depends on actions of others
- As a consequence, optimal decision depends on the decisions of others
- Other examples include: political candidate choosing a policy platform, bidder in an auction, negotiator in a purchase,...
- GT: framework to analyse decisions in presence of strategic interdependence

Chapter 5's plan

- (5.1): Monopoly and oligopoly pricing: motivation
- (5.2): Elements of a game and examples
- (5.3): Solving Games, Predicting Behaviour
- (5.4): Best-response function, and mixed strategies

5.2.- Elements of a Game and Examples

- Elements of a game
- Example 1: Prisoner's dilemma
- Example 2: Coordination game
- Example 3: Matching pennies
- Example 4: Stag hunt

Elements of a "Game"

- Players: who is involved?
- For each player, a set of actions: what can you play?
- Outcomes: for each set of actions, what happens?
- Payoffs: what are the preferences over these outcomes?

Example 1: Prisoner's Dilemma

- Two suspects held in separate cells, asked if they committed the crime
- If both confess, sentence of 5 years in prison each. If none does, 2 years each. If one confesses and other not, 1 and 10 years, respec.
- Players: the two suspects. Actions for each: Confess or Don't Confess. Payoffs: equal to (minus) the years in prison
- Representation in a table:

1\2	Don't Confess	Confess
Don't Confess	-2,-2	-10,-1
Confess	-1,-10	-5,-5

- Gains from cooperation but each has an incentive to "free-ride"
- Situations modelled similarly: working in a joint project, arms race,..

Example 2: "Battle of the Sexes"

- Two people wish to go out together but two concerts available
- One prefers Bach and other Stravinsky but equally dislike going alone
- Representation in a table:

1\2	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

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- Situations modelled similarly: two officials of a political party deciding the stand on an issue, two merging firms selecting one of the two current computer technologies
- Here, better to cooperate but disagreement on the best outcome

Example 3: Matching Pennies

- 2 persons choose simultaneously to show head or tail of a coin
- If they match, person 2 pays £1 to person 1 and if they do not match, person 1 pays £1 to person 2
- Representation in a table:

1\2	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

.

- Situations modelled similarly: choices for the appearance of a new product by an established producer and a new firm

Example 4: Stag Hunt

- 2 hunters choose to remain attentive to pursuit a stag or catch a hare
- If they both pursuit the stag, they catch it and share it equally
- If any hunter tries to catch a hare, the stag scapes and the hare belongs to the defeating hunter alone
- They prefer to catch the stag and share it equally than catch a hare
- Representation in a table:

1\2	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

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- Situations modelled similarly: security dilemma faced by a pair of countries

In Sum, Game Theory...

- Set of tools to analyse behaviour in presence of strategic interdependence
- For example, firms in an oligopoly market
- Static games: players play simultaneously and only once
- Further examples: "prisoner's dilemma", "battle of the sexes", "matching pennies", "stag hunt",...
- Representation in "normal form"

Chapter 5.3: Solving Games, Predicting Behaviour

- What should we expect players to play?
- Looking for reasonable concepts in simple predictable games and apply these concepts in other settings
- Here, static ("simultaneous move" or "strategic") games
- Solution concepts:
  - Use dominant strategies
  - Don't use dominated strategies
  - Play Nash equilibrium strategies
- Assume that players are rational:
  - each chooses her best action according to her preferences

Dominant Strategies

- "Prisoner's dilemma":

1\2	DC	C
DC	-2,-2	-10,-1
C	-1,-10	-5,-5

- What action would you choose?
- No matter what the other does, it is better to play "Confess":
  - If she plays DC, one obtains -1 by playing C and -2 by playing DC
  - If she plays C, one obtains -5 by playing C and -10 by playing DC
- Formally, "Confess" strictly dominates "Don't Confess"
- "Confess" is a dominant strategy or action
- Both confessing is the outcome! Conflict with Pareto-optimality

Introducing Notation

- $i = 1, ..., I$ the players of a game (e.g. in PD $i = 1, 2$)
- s_i a strategy or action for player i (e.g. $s_1 = C$ or $s_1 = DC$)
- $s = (s_1, ..., s_I)$ a strategy profile: one strategy for each player (e.g. $s = (C, DC)$ or $s = (CD, DC)$)
- $s = (s_i, s_{-i})$ a strategy profile, where $-i$: all the players except i
- $u_i(s) = u_i(s_1, ..., s_I)$ the payoff for player i if s is played (e.g. $u_1(C, DC) = -1$, $u_2(C, DC) = -10$,...)

Dominant Strategies

- Definition: Player i's strategy s_i'' strictly dominates strategy s_i' if

$$u_i(s_i'', s_{-i}) > u_i(s_i', s_{-i}) \text{ for any } s_{-i}$$

- Accordingly, we say that strategy s_i' is strictly dominated
- Example: strategy "Don't Confess" is strictly dominated by "Confess"
- Definition: A strictly dominant strategy for player i is a strategy that strictly dominates all her other strategies
- Example: "Don't Confess" is strictly dominant for both players
- Rational players play dominant strategies

Strictly Dominated Strategies

- Problem: strictly dominant strategies rarely exist. Examples:

(a)

1\2	L	R
U	1,-1	-1,-1
M	-1,1	1,-1
D	-2,5	-3,2

(b)

1\2	L	R
U	5,1	4,0
M	6,0	3,1
D	6,4	4,4

- However, a strictly dominated strategy may still exist.
- Example: strategy D in game (a)
- Rational players do not play dominated strategies

Weakly Dominated Strategies

- Definition: Player i's strategy s_i'' weakly dominates strategy s_i' if

$$u_i(s_i'', s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for any } s_{-i}$$
$$\text{with strict inequality for some } s_{-i}$$

- Example: U and M in (b) are weakly dominated
- Should we rule out weakly dominated strategies as well? No!
  - Playing M can be as good as playing D if 1 *believes* that 2 will play L

Nash Equilibrium

- Problem: strictly or weakly dominated strategies may not exist:

1\2	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

- Definition: A strategy profile $(s_1, s_2, ..., s_I)$ constitutes a Nash equilibrium if for every player i

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for every action } s_i'$$

- Here, we are assuming that...
(1) Players are rational (given the belief about others' actions)
(2) Their beliefs about the actions of the others are correct

Example 1: Prisoner's Dilemma

1\2	Don't Confess	Confess
Don't Confess	-2,-2	-10,-1
Confess	-1,-10	-5,-5

- (C, C) is a NE:
  - given that 2 plays C, playing C is better than DC for 1 (-5>-10)
  - given that 1 plays C, playing C is better than DC for 2 (-5>-10)
- (C, DC) is a not a NE:
  - given that 1 plays C, playing C is better than DC for 2 (-5>-10)
- (DC, C) is a not a NE:
  - given that 2 plays C, playing C is better than DC for 1 (-5>-10)
- (DC, DC) is a not a NE:
  - given that 2 plays DC, playing C is better than DC for 1 (-1>-2)

Remarks

- NE may not be unique!! Example 2: coordination Game (or BoS)

1\2	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

- NE: (Bach, Bach) and (Stravinsky, Stravinsky)
- NE may not exist! Example 3: Matching Pennies

1\2	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- NE: none!

Exercise

- Exercise: what are the NE in the Stag Hunt game (example 4)?

1\2	Stag	Hare
Stag	2,2	0,1
Hare	1,0	1,1

- Find them intuitively first and show it formally (using definition)

- So far, we've examined all combination profiles to find whether they were NE
- If there are many possible actions and many players, this might be difficult

5.3.- Best Response Functions

- What is the best action for a player to any given list of others' actions? BoS
  - for player 1, Bach is a best response if 2 chooses Bach
  - for player 1, Stravinsky is a best response if 2 chooses Stravinsky
- Best response to a given action of the others may not be unique. Example:

	L	M	R
T	1,1	1,0	0,1
B	1,0	0,1	1,0

- Formally, denote i's best response to actions of others a_{-i} as $B_i(a_{-i})$
  - $B_1(Bach) = Bach$ and $B_1(Stravinsky) = Stravinsky$ in Bos
  - $B_1(L) = \{T, B\}$, $B_1(M) = \{T\}$, $B_1(R) = \{B\}$ in previous game

Best Response Functions and Nash

- Remember that in a NE, no player can do better than playing her NE strategy if the others play their NE strategies
- Hence, we can redefine the concept of NE in terms of best responses
- Definition: $(s_1, ..., s_I)$ is a NE if and only if every player's action is a best response to the others' actions, i.e. s_i is in $B_i(s_{-i})$ for every i
- Method to find NE: (a) Find best response functions and (b) Find strategy profiles that are mutually best responses:

	L	C	R
T	1,2	2,1	1,0
M	2,1	0,1	0,0
B	0,1	0,0	1,2

- If best-resp. have 1 element: $(s_1, ., s_I)$ is a NE iff $B_i(s_{-i}) = s_i$ for every i
- This is a system of I equations with I unknowns $(s_i\text{'s})$

Example: A Synergistic Relationship

- 2 people in synergistic relationship: if both devote effort, both better off
- Suppose that the relationship value for i $(i = 1, 2)$ is given by $a_i(c + a_j - a_i)$, where a_i and a_j are own and other's efforts, resp.
- Players: $1, 2$. Strategies: $a_i > 0$ and payoffs: $a_i(c + a_j - a_i)$
- Best response: suppose player j plays a_j, what is my best response?
- Compute (partial) derivative and equate to 0 (check second order condition)

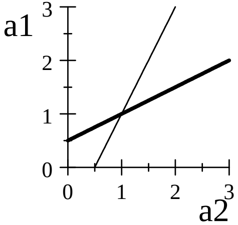
$$c + a_j - 2a_i^* = 0; \quad B_i(a_j) \equiv a_i^* = \frac{c + a_j}{2}$$

- Nash equilibrium: intersection of best reply functions, i.e. solve for

$$a_1^* = \frac{c + a_2^*}{2} \text{ and } a_2^* = \frac{c + a_1^*}{2}\text{; and therefore } a_1^* = a_2^* = c$$

Example: A Synergistic Relationship (2)

- Suppose that $c = 1$. Best responses are $B_1(a_2) = \frac{1+a_2}{2}$ and $B_2(a_1) = \frac{1+a_1}{2}$. NE is: $a_1^* = a_2^* = 1$. Representation:



- Best response functions: $B_1(a_2)$ (thick line, from horizontal to vertical axis) and $B_2(a_1)$ (thin line, from vertical to horizontal axis)
- NE: $(1, 1)$ (intersection of the best response functions)

Nash Equilibrium and Dominance

- A strictly dominated action is not a best response to any action
- Therefore, a strictly dominated action is not used in any NE
- One can eliminate strictly dominated actions when looking for NE
  e.g. in PD, DC is never part of a NE: only possible NE is (C,C)
- Can an action of a NE be weakly dominated? Yes!. Examples:

	B	C
B	1,1	0,0
C	0,0	0,0

	B	C
B	1,1	2,0
C	0,2	2,2

- C is weakly dominated in both games but (C,C) is a NE in both games
- (B,B) is a NE in both games. In the left game, this NE is better for both than the previous NE whereas in the right game, it is worse

Matching Pennies (MP)

- Some games do not have any NE. Example:

1\2	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

- But here, players may want to introduce random behaviour
- Other examples include government auditing taxpayers
- Can we find an equilibrium when randomisation is allowed?

Mixed Strategies

- $s_i \in S_i$ (deterministic) *pure* strategy and S_i set of pure strategies
- Definition:
  - A mixed strategy for player i, σ_i, assigns to each pure strategy s_i a probability $\sigma_i(s_i)$ that it will be played (where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$)
- In MP, (a) σ_i' such that $\sigma_i'(H) = \frac{1}{2}$, $\sigma_i'(T) = \frac{1}{2}$ or $\sigma_i' = (\frac{1}{2}, \frac{1}{2})$ (b) $\sigma_i'' = Head$, i.e. $\sigma_i''(H) = 1$, $\sigma_i''(T) = 0$ or $\sigma_i'' = (1, 0)$
- Definitions:
  - set of mixed strategies: $\Delta(S_i)$, strategy profile: $\sigma = (\sigma_1, ..., \sigma_I)$
  - $u_i(\sigma)$, i's expected payoff to mixed profile σ, $u_i(\sigma) \equiv E_\sigma[u_i(s)]$

Examples:

- $u_1(\sigma_1', \sigma_2') =$
  $\frac{1}{2}\frac{1}{2}u_1(H, H) + \frac{1}{2}\frac{1}{2}u_1(H, T) + \frac{1}{2}\frac{1}{2}u_1(T, H) + \frac{1}{2}\frac{1}{2}u_1(T, T) = 0$

- $u_1(\sigma_1', \sigma_2'') =$
  $\frac{1}{2}1u_1(H, H) + \frac{1}{2}0u_1(H, T) + \frac{1}{2}1u_1(T, H) + \frac{1}{2}0u_1(T, T) = 0$

- $u_1(\sigma_1'', \sigma_2'') =$
  $1 * 1u_1(H, H) + 1 * 0u_1(H, T) + 0 * 1u_1(T, H) +) * 0u_1(T, T) = 1$

Mixed Strategy Nash Equilibrium

- A game can be redefined (in normal form) as $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$
- Definition: A strategy profile $(\sigma_1, \sigma_2, ..., \sigma_I)$ constitutes a mixed strategy Nash equilibrium if for every $i = 1, ..., I$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}) \text{ for every } \sigma'_i \in \Delta(S_i)$$

- Example: (σ'_1, σ'_2) where $\sigma'_i = (\frac{1}{2}, \frac{1}{2})$ is a MSNE in MP
- But, is there any other?
- $(\sigma_1, ..., \sigma_I)$ is MSNE iff strategy σ_i is best response to σ_{-i} for all i
- Example: σ'_1 is a best response to σ'_2 and vice versa in MP

Pure and Mixed Strategy NE

- Example: another coordination game ("Meeting in New York"):

$1\backslash 2$	ES	GC
ES	100,100	0,0
GC	0,0	1000,1000

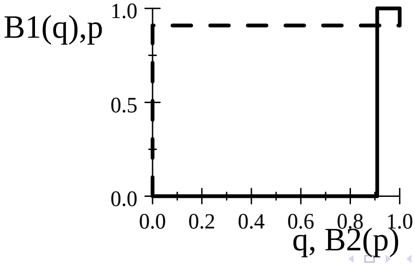
- (ES, ES) and (GC, GC) are two (pure strategy) Nash equilibrium
- Is there any mixed strategy Nash equilibrium? How do we find it?

A Method to Find Pure and Mixed Strategy NE

1. Find best-response functions. For each $[(p, 1-p), (q, 1-q)]$

$$B_1(q) = \begin{cases} 0 \text{ if } q \in [0, \frac{10}{11}) \\ [0,1] \text{ if } q = \frac{10}{11} \\ 1 \text{ if } q \in (\frac{10}{11}, 1] \end{cases} \text{ and } B_2(p) = \begin{cases} 0 \text{ if } p \in [0, \frac{10}{11}) \\ [0,1] \text{ if } p = \frac{10}{11} \\ 1 \text{ if } p \in (\frac{10}{11}, 1] \end{cases}$$

2. For 2-player-2-action games, represent and find intersections



A Method to Find Pure and Mixed Strategy NE (2)

- Or, assume $[(p^*, 1-p^*), (q^*, 1-q^*)]$ is MSNE and find conditions:
  - If $p^* = 0$ then $q^* = 0$. If $q^* = 0$ then $p^* = 0$: $[(0,1),(0,1)]$ is a NE
  - Similarly $[(1,0),(1,0)]$ is a NE
  - If $0 < p^* < 1$ then $q^* = \frac{10}{11}$ and then since $0 < q^* < 1$, $p^* = \frac{10}{11}$
  - Hence, $[(\frac{10}{11}, \frac{1}{11}), (\frac{10}{11}, \frac{1}{11})]$ is a NE
- Exercise: show that (σ_1', σ_2') is the unique MSNE in MP

- Notice that in the strictly MSNE, each player is indifferent among all the pure strategies played with positive probability
- This is a general property, see Osborne (2004)

- Proposition: Every static game with finitely many actions has a MSNE