# Lecture 4: Simultaneous Move Games Theory (3)

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# Matching Pennies (MP)

- Some games do not have any NE. Example:

| 1\2 | Head | Tail |
|---|---|---|
| Head | 1,-1 | -1,1 |
| Tail | -1,1 | 1,-1 |

- But here, players may want to introduce random behaviour
- Other examples include government auditing taxpayers
- Can we find an equilibrium when randomisation is allowed?

# Mixed Strategies

- Notation: $s_i \in S_i$ (deterministic) *pure* strategy and $S_i$ set of pure strategies
- Definition: A mixed strategy for player $i$, $\sigma_i$, assigns to each pure strategy $s_i$ a probability $\sigma_i(s_i)$ that it will be played (where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$)
- Examples: in MP, (a) $\sigma_i'$ such that $\sigma_i'(H) = \frac{1}{2}$, $\sigma_i'(T) = \frac{1}{2}$ or $\sigma_i' = (\frac{1}{2}, \frac{1}{2})$ (b) $\sigma_i'' = Head$ , i.e. $\sigma_i''(H) = 1$, $\sigma_i''(T) = 0$ or $\sigma_i'' = (1, 0)$
- Definitions: set of mixed strategies: $\Delta(S_i)$, strategy profile: $\sigma = (\sigma_1, ..., \sigma_I)$.
- $u_i(\sigma)$, $i$'s expected payoff to the mixed strategy profile $\sigma$, $u_i(\sigma) \equiv E_\sigma[u_i(s)]$
- Examples:
  $u_1(\sigma_1', \sigma_2') = \frac{1}{2}\frac{1}{2}u_1(H, H) + \frac{1}{2}\frac{1}{2}u_1(H, T) + \frac{1}{2}\frac{1}{2}u_1(T, H) + \frac{1}{2}\frac{1}{2}u_1(T, T) = 0$
  $u_1(\sigma_1', \sigma_2'') = \frac{1}{2}1u_1(H, H) + \frac{1}{2}0u_1(H, T) + \frac{1}{2}1u_1(T, H) + \frac{1}{2}0u_1(T, T) = 0$
  $u_1(\sigma_1'', \sigma_2'') = 1*1u_1(H, H)+1*0u_1(H, T)+0*1u_1(T, H)+0*0u_1(T, T) = 1$

# Mixed Strategy Nash Equilibrium

- We can redefine again a game (in normal form) as $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$
- Definition: A strategy profile $(\sigma_1, \sigma_2, ..., \sigma_I)$ constitutes a mixed strategy Nash equilibrium if for every $i = 1, ..., I$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for every } \sigma_i' \in \Delta(S_i)$$

- Example: $(\sigma_1', \sigma_2')$ where $\sigma_i' = (\frac{1}{2}, \frac{1}{2})$ is a MSNE in MP
- But, is there any other?
- Alternative definition of NE: $(\sigma_1, \sigma_2, ..., \sigma_I)$ is a MSNE iff strategy $\sigma_i$ is a best response to $\sigma_{-i}$ for all $i$
- Example: $\sigma_1'$ is a best response to $\sigma_2'$ and vice versa in MP

# Pure and Mixed Strategy NE

- Example: another coordination game ("Meeting in New York"):

| $1\backslash 2$ | $ES$ | $GC$ |
|---|---|---|
| $ES$ | 100,100 | 0,0 |
| $GC$ | 0,0 | 1000,1000 |

- $(ES, ES)$ and $(GC, GC)$ are two (pure strategy) Nash equilibrium

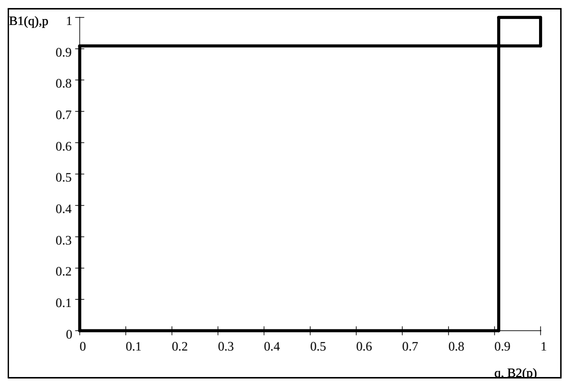
- Is there any mixed strategy Nash equilibrium as well? How do we find it?

# A Method to Find Pure and Mixed Strategy NE

1. Find best-response functions. For each $[(p, 1-p), (q, 1-q)]$

$$B_1(q) = \begin{cases} 0 \text{ if } q \in [0, \frac{10}{11}) \\ [0,1] \text{ if } q = \frac{10}{11} \\ 1 \text{ if } q \in (\frac{10}{11}, 1] \end{cases} \text{ and } B_2(p) = \begin{cases} 0 \text{ if } p \in [0, \frac{10}{11}) \\ [0,1] \text{ if } p = \frac{10}{11} \\ 1 \text{ if } p \in (\frac{10}{11}, 1] \end{cases}$$

2. For two player-two action games, represent them and find intersections



# A Method to Find Pure and Mixed Strategy NE (2)

2. Or, assume that $[(p^*, 1-p^*), (q^*, 1-q^*)]$ is a MSNE and look for conditions:

   If $p^* = 0$ then $q^* = 0$. If $q^* = 0$ then $p^* = 0$. Hence $[(0,1),(0,1)]$ is a NE

   Similarly $[(1,0),(1,0)]$ is a NE.

   If $0 < p^* < 1$ then $q^* = \frac{10}{11}$ and then since $0 < q^* < 1$, $p^* = \frac{10}{11}$

   Hence, $[(\frac{10}{11}, \frac{1}{11}), (\frac{10}{11}, \frac{1}{11})]$ is a NE

- Exercise: show that $(\sigma_1', \sigma_2')$ is the unique MSNE in MP
- Notice that in the strictly MSNE, each player is indifferent among all the pure strategies played with positive probability
- This is a general property, see Osborne (2004)
- Proposition (Existence): Every static game in which each player has finitely many actions has a MSNE