Lecture 7: Dynamic Games of Incomplete Information

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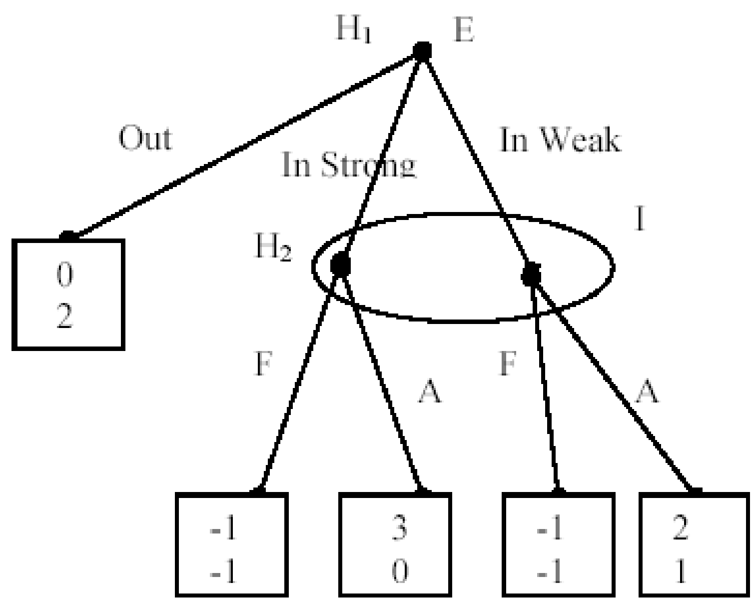
Today's Lecture

- In dynamic games of incomplete info, BNE and SPNE may not be strong enough
- Approach: combine two previous methods
  a) Transform extensive game of incomplete info into game of imperfect info
  b) Represent the game in normal form and look for BNE
  c) Refine the prediction by looking for SPNE
  d) Not strong enough!
- Example: entry game (starting already from point b)
- Refining the NE: Weak Perfect Bayesian Equilibrium (WPBE)
- Problem: may not be SPNE
- Refining WPBE and SPNE: Perfect Bayesian Equilibrium
- Refining PBE: Sequential Equilibrium
- Sometimes WPBE is also called Weak Sequential Equilibrium

Example 1



- Normal form:

$E\backslash I$	F	A
Out	0 , 2	0,2
InS	-1,-1	3,0
InW	-1,-1	2,1

- NE: (Out, F), (InS, A). SPNE: . Plausible?

- Sequentially rationality principle: play should be optimal for some belief (about type of entry)

- Idea of WPBE: define beliefs and apply sequential rationality

Formal Definitions: Beliefs

- Remember that:
  χ is the set of nodes
  $s(x)$ are the set of successors of node x
  Ξ is the collection of information sets (partition of χ)
  $H(x)$ is the information set containing node x
  $\cup_{x \in H} s(x)$ all the successors at information set H
  $\iota(H)$ determines the playing player at info set H

- Definition: A *system of beliefs* μ is a mapping $\mu : \chi \rightarrow [0,1]$ such that $\sum_{x \in H} \mu(x) = 1$ for any $H \in \Xi$.

- In previous example: (1) in H_1 and $(\frac{2}{3}, \frac{1}{3})$ in H_2.

- Interpretation: probability assessment of likelihood of each node at each info set

Sequential Rationality: Restrictions on Strategies

- Definition: A behaviour strategy profile δ is *sequentially rational* given a system of beliefs μ if, for all H, the actions of $\iota(H)$ at $H \cup (\cup_{x \in H} s(x))$ are optimal given an initial probability over H defined by μ, and given that all the other players adhere to δ_{-i}.
- Given beliefs $\left[1, \left(\frac{2}{3}, \frac{1}{3}\right)\right]$, the only sequentially rational strategy profile is (InS, A)
  a) at H_2 (I playing), the payoff from A is $\frac{2}{3} * 0 + \frac{1}{3} * 1 = \frac{1}{3}$
  whereas the payoff from F is $\frac{2}{3} * (-1) + \frac{1}{3} * (-1) = -1$
  b) at H_1 (E playing and given $\delta_{-i} = A$),
  the payoff from Out is 0, from InS is 3 and from InW is 2
- Can you see why, given beliefs $\left[1, \left(\frac{2}{3}, \frac{1}{3}\right)\right]$, (Out, F) is not sequentially rational?
- In fact F is never going to be a sequentially rational strategy (for any belief)!
- See MWG book for a definition in pure and mixed strategies

Restrictions on Beliefs

- Beliefs should be "consistent" with the strategy profile (whenever possible)
- Example (1): suppose that E plays $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$ (=($p(Out), p(InS), p(InW)$))
  What should be the beliefs of I on H_2? Exactly, $\left(\frac{2}{3}, \frac{1}{3}\right)$
- Example (2): suppose that E plays $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$
  What should be the beliefs of I on H_2 now? $(1, 0)$
- Formally, we use Bayes rule. For a strategy profile δ^* and for any $x \in H$

$$\mu^*(x) = \frac{\Pr ob(x \mid \delta^*)}{\Pr ob(H \mid \delta^*)}$$

- Example (3): suppose that E plays $(1, 0, 0)$
  What should be the beliefs of I? Bayes rule can't be used ($\Pr ob(H \mid \delta^*) = 0$)
  In the WPBE, we will assign one belief (any but one needs to be assigned!)

Weak Perfect Bayesian Equilibrium

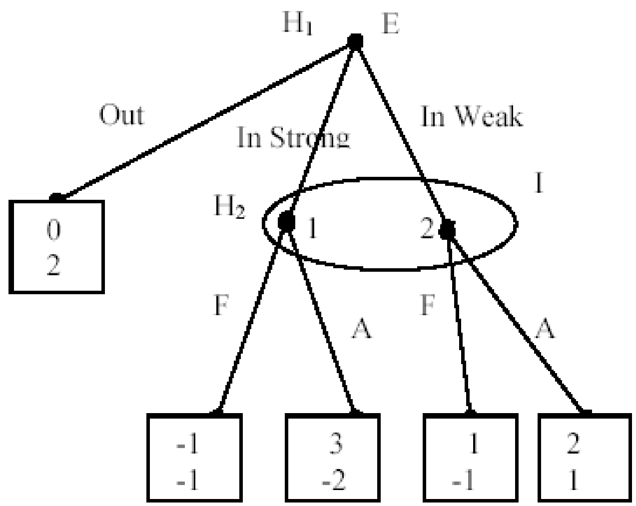
- Definition: (δ^*, μ^*) is a *Weak Perfect Bayesian equilibrium* iff
  a) the behaviour strategy profile δ^* is sequentially rational given μ^*, and
  b) wherever possible μ^* is computed from δ^* using Bayes rule. That is for any information set H such that $\Pr ob(H \mid \delta^*) > 0$, we have that, for any $x \in H$,

$$\mu^*(x) = \frac{\Pr ob(x \mid \delta^*)}{\Pr ob(H \mid \delta^*)}.$$

- Example: $(\delta^*, \mu^*) = [[InS, A], [1, (1, 0)]]$ is the only WPBE

- Remarks:
  Nash equilibrium requires sequential rationality only in H s.t. $\Pr ob(H \mid \delta^*) > 0$
  WPBE also requires specification of (any) beliefs and an optimal action in H such that $\Pr ob(H \mid \delta^*) = 0$

Example 2



- Now I is willing to play F if she knew that InS has been played

Suppose that $[[(p_{Out}, p_{InS}, p_{InW}), (p_F, p_A)], [1, (\mu_1, \mu_2)]]$ is a WPBE
Given the system of beliefs, the optimal decision at H_2 is
Play F iff $\mu_1(-1) + (1 - \mu_1)(-1) \geq \mu_1(-2) + (1 - \mu_1)(1)$ or iff $\mu_1 \geq \frac{2}{3}$

- Suppose that $\mu_1 > \frac{2}{3}$, then the sequentially rational decision at H_2 is $p_F = 1$ and therefore $p_A = 0$
  Then, E's sequentially rational strategy at H_1 given the strategy of I is $p_{InW} = 1, p_{Out} = p_{InS} = 0$ since they payoffs are 1, -1 and 0, resp.
  Then, by Bayes rule, $\mu_1 = 0$ (and $\mu_2 = 1$). Contradiction!!

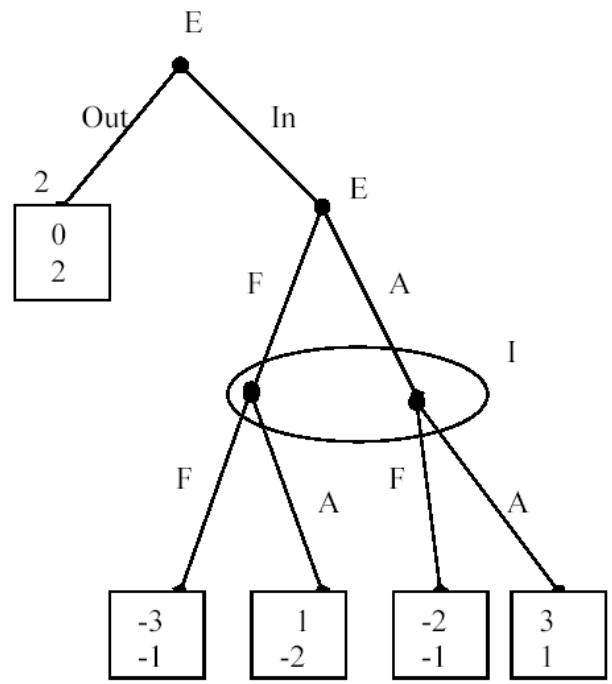
- Suppose that $\mu_1 < \frac{2}{3}$, then the sequentially rational decision at H_2 is $p_F = 0$ and therefore $p_A = 1$
  Then, E's sequentially rational strategy at H_1 given the strategy of I is $p_{InS} = 1, p_{Out} = p_{InW} = 0$ since they payoffs are 3, 0 and 2, resp.
  Then, by Bayes rule, $\mu_1 = 1$ and $\mu_2 = 0$. Contradiction!!

- Suppose that $\mu_1 = \frac{2}{3}$, then (a) assume that either $p_{InS} > 0$ or $p_{InW} > 0$
  Can apply Bayes and since $\mu_1 = \frac{2}{3}$, $p_{InS} = 2p_{InW}$ & $p_{InS} > 0$ and $p_{InW} > 0$
  Then player I should be indifferent between InS and InW:
  $p_F(-1) + (1 - p_F)(3) = p_F(1) + (1 - p_F)(2)$ or $p_F = \frac{1}{3}$ (and $p_A = \frac{2}{3}$)
  Then payoffs for E for InS (same for InW) are $\frac{1}{3}(-1) + \frac{2}{3}(3) = \frac{5}{3}$ whereas for Out are 0
  Therefore $p_{Out} = 0$ and then $p_{InS} = \frac{2}{3}$ and $p_{InW} = \frac{1}{3}$
  $\left[\left[\left(0, \frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right], \left[1, \left(\frac{2}{3}, \frac{1}{3}\right)\right]\right]$ is a WPBE

- Suppose that $\mu_1 = \frac{2}{3}$ and (b) assume that $p_{InS} = 0$ and $p_{InW} = 0$ (then $p_{Out} = 1$)
  But then, given (p_F, p_A), E's payoff for playing Out should be, at least, as large as for playing InW (and InS as well)
  i.e. $0 \geq p_F(1) + (1 - p_F)(2)$ or $p_F \geq 2$. Contradiction!!!

Example 3: WPBE may not be SPNE

- Problem: beliefs only need to be consistent in the equilibrium path
- Example from Lecture 4:



- Normal form:

E\I	A if In	F if In
Out, A if In	0,2	0,2
Out, F if In	0,2	0,2
In, A if In	3, 1	-2,-1
In, F if In	1,-2	-3,-1

- NE: $[(In, A$ if $In), A$ if $In]$ and $[(Out, A$ if $In), F$ if $In]$. SPNE: only the first

- May the second be supported as a WPNE? If the beliefs $[1, 1, (1, 0)]$ are added:
  Consistent beliefs when possible? Yes, because last info set is not reached in eq.
  Sequentially rational given beliefs and other's strategy? Yes, because...
  at the last info set, given $(1, 0)$ beliefs, F is sequentially rational
  at the before-to-last node, given I's strategy (F), A is sequentially rational
  at the first node, given I's strategy (F), $(Out, A$ if $In)$ is sequentially rational

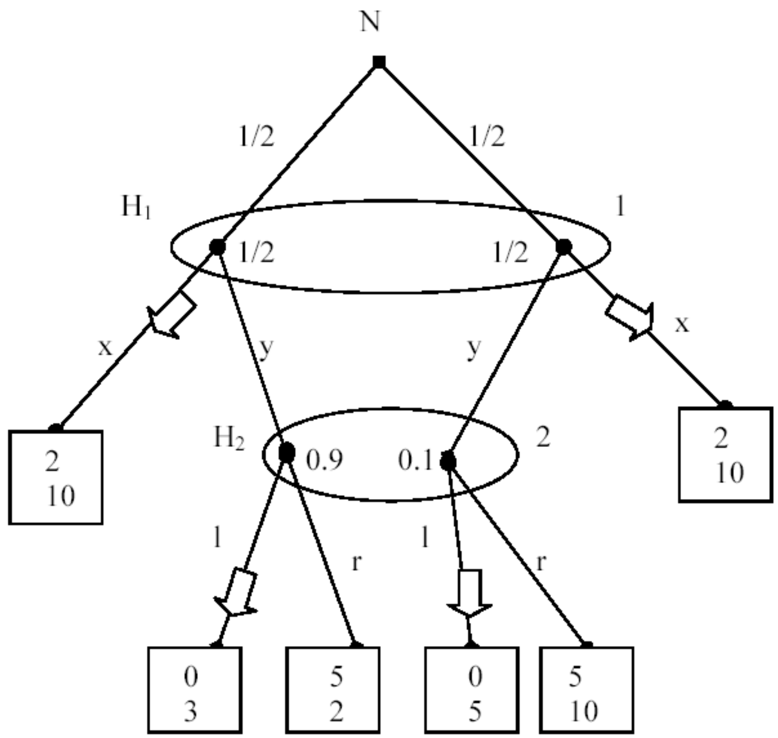
Perfect Bayesian Equilibrium

- Definition: A *Perfect Bayesian Equilibrium* (PBE) is a WPBE that induces a WPBE in every subgame

- Previous WPBE: $([(Out, A \text{ if } In), F \text{ if } In], [1, 1, (1, 0)])$ is not a PBE
  At the subgame starting at the 2nd node $([(A \text{ if } In), F \text{ if } In], [1, (1, 0)])$ is not a WPBE since...
  Beliefs at the last info set are not consistent with strategy profile

- However, PBE does not place much restrictions on out-of-equilibrium beliefs

Example 4: PBE may still not be strong enough



- The strategy profile and the specified beliefs form a WPBE:
  Beliefs are consistent (when possible): at H_1 they are and at H_2 they are free
  Given beliefs at H_2, sequentially rational at H_2 since

$$0.9 * 3 + 0.1 * 5 = 3.2 > 2.8 = 0.9 * 2 + 0.1 * 10$$

  Given beliefs at H_1 and 2's strategy, sequentially rational at H_1 since

$$0.5 * 2 + 0.5 * 2 = 2 > 0 = 0.5 * 0 + 0.5 * 0$$

- Given that the only subgame is the whole game, this WPBE is also a PBE
- Are player 2 beliefs "sensible"?
- Given that Nature assigns 0.5 to each branch and 1's choice cannot depend on Nature's outcome, the only sensible beliefs for 2 seem to be $(0.5, 0.5)$

Sequential Equilibrium

- Idea: Make out-of-equilibrium beliefs consistent with strategies that reach all info sets and aren't very different from equilibrium strategies
- Definition: A behaviour strategy is *strictly mixed* if every action at each info set is selected with strictly positive probability
- Note: If all the strategies of the strategy profile are strictly mixed, then all info sets are reached and Bayes rule can always be used
- Definition: (δ, μ) is *consistent* iff there exists a sequence of strictly mixed behavior strategy profiles $\delta_n \longrightarrow \delta$ such that $\mu_n \longrightarrow \mu$, where μ_n is derived from δ_n using Bayes rule
- Example: in the previous graph, the pair of strategy and beliefs was not consistent Beliefs $(0.5, 0.5)$ at H_2 would have been consistent with the strategy
- Definition: (δ^*, μ^*) is a *sequential equilibrium* (SE) iff δ^* is sequentially rational given μ^* and (δ^*, μ^*) is consistent

- Remark 1: SE places additional restrictions on the beliefs of a WPBE
- Remark 2: any SE is subgame perfect
- Note:

$$SE \subseteq PBE \subseteq \left\{ \begin{array}{c} WPBE = WSE \\ SPNE \end{array} \right\} \subseteq BNE$$

- Exercise: check that $[(Out, A$ if $In), F$ if $In]$ in Example 3 is not a SE