Lecture 6: Static Games of Incomplete Information

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Today's Lecture

- Game of incomplete information: "some do not know the payoff of other players"
  e.g. firms may not know the costs of the other firms
  e.g. bidders may not know the valuations of the others in an auction
- Problem: what are the beliefs of the others?
  e.g. firms do not know what the others think about their own costs
- Further problems: what are the beliefs about the beliefs and so on?
- Approach (Harsanyi, 1967-68): assume that...
  a) unknown parameter of a player's payoff is realisation of a random variable,
  b) only the player (not the others) observes the realisation (the "type"), but
  c) all players (including herself) know the distribution of the random variable (and this is common knowledge)
- Hence, transformation of a game of incomplete information into one of imperfect information (Bayesian game)

Example

- Prisoner 1 does not know whether prisoner 2 is a "foe" or a "friend"
  Prisoner 2 knows, of course, who she is

- Payoffs are (a) if Prisoner 2 is a "foe" and (b) is she is a "friend"

(a)

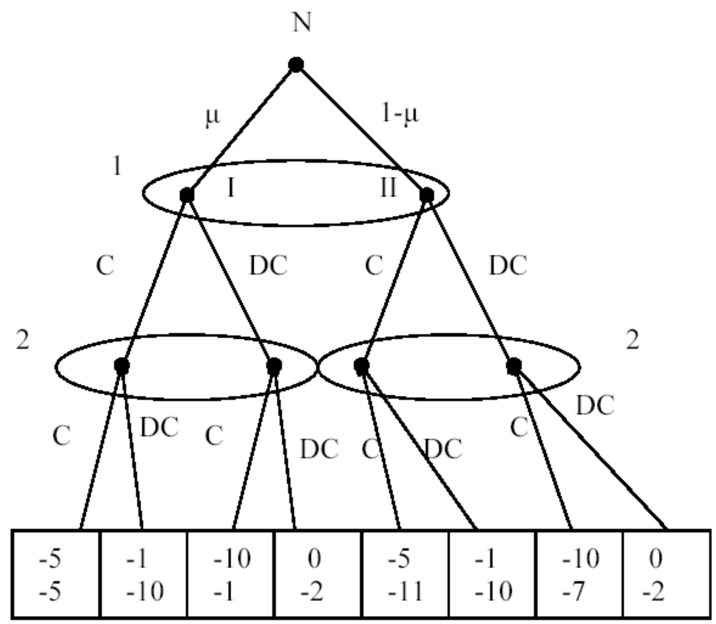
$1\backslash 2$	DC	C
DC	0, -2	-10, -1
C	-1, -10	-5, -5

(b)

$1\backslash 2$	DC	C
DC	0, -2	-10, -7
C	-1, -10	-5, -11

- Problem: Prisoner 2 does not know what Prisoner 1 thinks, and Prisoner 1 does not know what Prisoner 2 thinks about what she (Prisoner 1) thinks

- Harsanyi solution: assume that $\Pr ob(type\ I) = \mu$ and $\Pr ob(type\ II) = 1 - \mu$ and that this is common knowledge
  Now 2 knows what 1 thinks and 1 knows what 2 thinks about what she thinks,...



- Pure strategies for Prisoner 1:
  For Prisoner 2

Formal Definitions

- Bayesian game: $[I, \{S_i\}, \{u_i\}, \Theta, F()]$
  Payoff functions: $u_i(s_i, s_{-i}, \theta_i)$, where θ_i the type of i is only known to i
  Type space: $\Theta = \Theta_1 \times ... \times \Theta_I$
  Distribution of types: $F(\theta_1, ..., \theta_I)$ (common knowledge)

- Alternatively, one can view this as an "extended game"
  Nature selects types
  Players observe their own types, but not those of the others
  Players simultaneously select a pure strategy

- Strategy: mapping $\sigma_i : \Theta_i \longrightarrow S_i$, i.e., for any θ_i select $\sigma_i(\theta_i) \in S_i$

- Set of strategies Σ_i and the set of the strategy profiles $\Sigma = \Sigma_1 \times ... \times \Sigma_I$

- Given u_i and F, we can compute $\widetilde{u_i}(\sigma) = E_\theta \left[u_i(\sigma_1(\theta_1), ..., \sigma_I(\theta_I), \theta_i)\right]$

- Definition: A (pure strategy) "Bayesian Nash Equilibrium" of the Bayesian game $[I, \{S_i\}, \{u_i\}, \Theta, F()]$ is a (pure strategy) NE of the game $[I, \{\Sigma_i\}, \{\widetilde{u_i}\}]$

- Problem: difficult to find the NE strategy profiles (a strategy is a function)

- Equivalent definition (assume for simplicity Θ_i finite): A collection of decision rules $(\sigma_1, ..., \sigma_I)$ is a (pure strategy) Bayesian Nash equilibrium for the Bayesian game $[I, \{S_i\}, \{u_i\}, \Theta, F()]$, if and only if, for all i and all $\theta_i \in \Theta_i$ occurring with positive probability

$$E_{\theta_{-i}}\left[u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i) \mid \theta_i\right] \geq E_{\theta_{-i}}\left[u_i(s_i', \sigma_{-i}(\theta_{-i}), \theta_i) \mid \theta_i\right]$$
$$\text{for any } s_i' \in S_i$$

BNE (Example 1)

- Prisoner 2: C dominant strategy if type I and DC dominant strategy if type II
- Hence, if $(\sigma_1, \sigma_2(I), \sigma_2(II))$ is a BNE, then $\sigma_2(I) = C$ and $\sigma_2(II) = DC$
- Prisoner 1 has only one type. Play C whenever

$$E_{\theta_2}[u_1(C, \sigma_2(\theta_2))] \geq E_{\theta_2}[u_1(DC, \sigma_2(\theta_2))]$$

Taking expectations on the LHS,

$$\begin{aligned} & \mu\,[u_1(C, \sigma_2(I) \mid \theta_2 = I)] + (1-\mu)\,[u_1(C, \sigma_2(II) \mid \theta_2 = II)] \\ = \;& \mu\,[u_1(C, C \mid \theta_2 = I)] + (1-\mu)\,[u_1(C, DC \mid \theta_2 = II)] \\ = \;& \mu(-5) + (1-\mu)\,(-1) = -4\mu - 1 \end{aligned}$$

Similarly on the RHS,

$$\begin{aligned} & \mu\left[u_1(DC, \sigma_2(I) \mid \theta_2 = I)\right] + (1-\mu)\left[u_1(DC, \sigma_2(II) \mid \theta_2 = II)\right] \\ = \; & \mu(-10) + (1-\mu)(0) = -10\mu \end{aligned}$$

Hence play C, whenever $-4\mu - 1 \geq -10\mu$ or $\mu \geq \frac{1}{6}$

- Summarising, BNE is $(\sigma_1, \sigma_2(I), \sigma_2(II)) = \begin{cases} (C, C, DC) \text{ if } \mu \geq \frac{1}{6} \\ (DC, C, DC) \text{ if } \mu \leq \frac{1}{6} \end{cases}$
- Confess if likely that Prisoner 2 is a "foe"

Extension to Mixed Strategy BNE (Example 2)

- Payoffs are (a) if 1 is of type I and (b) if 1 is of type II and $Prob(\theta_1 = I) = p$ (where $p \leq \frac{1}{2}$) (Exercise: what would happen if $p > \frac{1}{2}$?)

(a)

1\2	L	R
U	0, -1	2, 0
D	2, 1	3, 0

(b)

1\2	L	R
U	1.5, -1	3.5, 0
D	2, 1	3, 0

- Assume that (z, x, y) is a Mixed Strategy BNE, where $z = Prob(U \mid \theta_1 = I)$, $x = Prob(U \mid \theta_1 = II)$ and $y = Prob(L)$

- Since playing D is dominant for 1 when she is of type I then, $z = 0$

- Player 1 will play U if she is of type II iff

$$y(1.5) + (1 - y)(3.5) \geq y(2) + (1 - y)(3) \text{ or iff } y \leq \frac{1}{2}$$

- Player 2 will play L iff

$$p(1)+(1-p)\left[x(-1)+(1-x)(1)\right] \geq 0 \text{ or iff } x \leq \frac{1}{2(1-p)} (\leq 1 \text{ by assumption})$$

$$b_1(p) = \begin{cases} 1 \text{ if } y \in [0, \frac{1}{2}) \\ [0,1] \text{ if } y = \frac{1}{2} \\ 0 \text{ if } y \in (\frac{1}{2}, 1] \end{cases} \text{ and } b_2(x,p) = \begin{cases} 1 \text{ if } x \in [0, \frac{1}{2(1-p)}) \\ [0,1] \text{ if } x = \frac{1}{2(1-p)} \\ 0 \text{ if } x \in (\frac{1}{2(1-p)}, 1] \end{cases}$$

- Suppose that $x = 0$, then, from $b_2(x,p)$, $y = 1$. If $y = 1$ then, from $b_1(x,p)$, $x = 0$. Hence $(0,0,1)$ is a BNE
  Suppose that $x = 1$, then, from $b_2(x,p)$, $y = 0$. If $y = 0$ then, from $b_1(x,p)$, $x = 1$. Hence $(0,1,0)$ is a BNE
  Suppose that $0 < x < 1$ then, from $b_1(x,p)$, $y = \frac{1}{2}$. If $y = \frac{1}{2}$ then, from $b_2(x,p)$, $x = \frac{1}{2(1-p)}$. Hence $(0, \frac{1}{2(1-p)}, \frac{1}{2})$ is a BNE

Extension to Continuous Strategy Space BNE (Example 3)

- Two firms are in a joint venture, sharing new products invented by any of them
  New product ("zigger") can be developed at cost $c \in (0,1)$
  Firms value it at θ_i^2 but the parameter $\theta_i \in [0,1]$ is unknown to the other firm
  When will each of the firms develop the zigger?

- Harsanyi: assume that $\theta_i \sim iidU[0,1]$ and that this is common knowledge

- Strategy: mapping $\sigma_i : \Theta_i = [0,1] \to S_i = \{0,1\}$, ($1$ *develop* and 0 *not dev*)

- Payoffs: for any θ_i
  $\theta_i^2 - c = E_{\theta_j}\left[u_i(1, \sigma_j(\theta_j), \theta_i) \mid \theta_i\right]$ if i develops
  $\theta_i^2(\Pr ob(\sigma_j(\theta_j) = 1)) = E_{\theta_j}\left[u_i(0, \sigma_j(\theta_j), \theta_i) \mid \theta_i\right]$ if i does not develop

- Hence, develop iff $\theta_i \geq \left[\frac{c}{1-\Pr ob(\sigma_j(\theta_j)=1)}\right]^{1/2}$

- For any strategy of j player i's best response is a "cutoff strategy":
  $\sigma_i(\theta_i) = \begin{cases} 1 \text{ iff } \theta_i \geq \theta_i^* \\ 0 \text{ iff } \theta_i < \theta_i^* \end{cases}$ (strategy characterised by some θ_i^*)

- Any NE is going to be of the form (θ_1^*, θ_2^*). Suppose that this is a BNE. Then: (exercise: show that $\theta_i^* > 0$)

$$\Pr ob(\sigma_j(\theta_j) = 1) = \int_{\theta_j^*}^{1} d\theta_j = 1 - \theta_j^* \text{ and } \theta_i^* = \left[\frac{C}{1-(1-\theta_j^*)}\right]^{1/2} \text{ or } (\theta_i^*)^2 \theta_j^* = c$$

- Similarly $\left(\theta_j^*\right)^2 \theta_i^* = c$ and therefore $\theta_i^* = \theta_j^* = c^{1/3}$

- Probability of none developing $(\theta_i^*)^2 = c^{2/3}$
  Probability of only one developing $2\theta_i^*(1 - \theta_i^*) =$
  Probability of none developing $(1 - \theta_i^*)^2 =$