Lecture 5: Repeated Games

Albert Banal-Estanol

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Cooperation?

- Example: prisoner's dilemma, again

$1\backslash 2$	DC	C
DC	3,3	0,4
C	4,0	1,1

- Dominated strategies:
- Nash equilibria:
- Do you expect players to cooperate (playing Don't Confess)?
- Maybe yes. Possible reason: players may expect to interact more in the future
- Simple way of modelling it: repeating the game!
- Two possibilities: (i) infinite repetitions (ii) finite repetitions

Today's Lecture

- Motivation: e.g. firms competing in the same environment for several periods
- Preliminaries for an infinitely repeated game

  How to evaluate infinite sequences of payoffs? Average and discounted criterion

  How do we define a strategy?
- Finding Nash Equilibria in infinitely repeated games:

  Players use the average payoff criterion

  Players use the discounted criterion
- Are these NE SPNE as well?
- Finitely repeated games

Preliminaries

- Definitions:

  Stage game: game played each period

  Supergame: dynamic game formed by infinite repetitions of the stage game

- Payoffs: need direct mapping from strategy profiles to payoffs (no terminal nodes)

  Distribute payoffs at the end of each stage game $v_i(t)$ (period t)

  For each profile, we have a path and sequence of payoffs $v_i = (v_i(1), v_i(2), ...)$

  To evaluate sequences of payoffs, one can use:

  (i) discounted payoffs: $u_i(v_i) = \sum_{t=1}^{\infty} \delta^{t-1} v_i(t)$

  (ii) average payoffs: $u_i(v_i) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} v_i(t)$

- Additional notation:
  $a_i(t)$ the action of player i in time t
  $a(t) = (a_1(t), ..., a_I(t))$ the action profile at time t
  $h(t) = (a(1), ..., a(t-1))$ the history up to (but not including) time t
  $h(1) = \varnothing$
  Examples of 3-histories in PD:
  $h(3) = ((C, C), (DC, C))$; $h(3) = ((DC, DC), (C, C))$; ...

- Strategies:
  Each $h(t)$ corresponds to an information set for each player
  Strategy: mapping from all t and all possible $h(t)$ to actions of stage game
  Examples of strategies in PD:
  (a) $\sigma_i(t, h(t)) = C$ for all t, $h(t)$
  (b) $\sigma_i(t, h(t)) = \begin{cases} C \text{ if } t > 1 \text{ and } h(t) \text{ contains a } C \\ DC \text{ otherwise} \end{cases}$

A Nash Equilibrium with Average Criterion

- Denote $\sigma_i^N(t, h(t))$ the strategy (a) in the previous slide
- Claim: If players use the average criterion, then $\left(\sigma_1^N, \sigma_2^N\right)$ is a Nash Equilibrium
- (Informal) proof: is it profitable to deviate, keeping other's strategy fixed?
  a) Utility from playing this strategy: 1 (sequence of 1's, average of 1 for any T)
  b) Utility from deviating: lower than (or equal to) 1 (sequence of 1's and 0's, average lower than 1)
- Hence, repeating the NE of the stage game is a NE of the supergame (also holds when using discounting criterion)

Another Nash Equilibrium with Average Criterion

- Take strategy (b) above:

$$\sigma_i(t, h(t)) = \begin{cases} C \text{ if } t > 1 \text{ and } h(t) \text{ contains a } C \\ DC \text{ otherwise} \end{cases}$$

- What would be the path if the two players use this strategy?

- Claim: When using average criterion, both players playing this strategy is a NE

- Proof: proceeding as before...

  a) Utility from playing this strategy: 3 (sequence of 3's, average of 3 for any T)

  b) Utility from (relevant) deviation: lower than (or equal to) 1

  Consider a deviation of player i to another strategy:

  Let t' be the first period in which the strategy dictates to play C when $h(t)$ contains only DC.

Sequence of payoffs:

$$v_i(t) = \begin{cases} 3 \text{ for } t < t' \\ 4 \text{ for } t = t' \\ \leq 1 \text{ for } t > t' \end{cases}$$

Note: if there is no such t', deviation has no effect and the average payoff is 3 ("not relevant")

- Hence, cooperation is possible! (sustained by the threat of punishment!)

More Nash Equilibrium with Average Criterion

- Problem: there are many more NE.
- Define $h^*(t)$ as follows. $h^*(1) = \varnothing$ and for any $t > 1$,

$$h^*(t) = \begin{cases} a_1(t') = C \text{ if } t' < t \text{ and } t' \text{ odd} \\ a_1(t') = DC \text{ if } t' < t \text{ and } t' \text{ even} \\ a_2(t') = DC \text{ if } t' < t \end{cases}$$

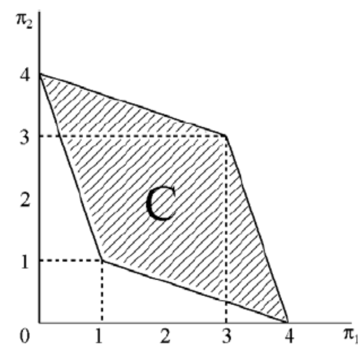
- This generates $h^*(\infty)$. Consider:

$$\sigma_1(t, h(t)) = \begin{cases} DC \text{ if } h(t) = h^*(t) \text{ and } t \text{ even} \\ C \text{ otherwise} \end{cases} \qquad \sigma_2 = \begin{cases} DC \text{ if } h(t) = h^*(t) \\ C \text{ otherwise} \end{cases}$$

- What would be the path if the two players use this strategy?
- What would be the sequence of payoffs? And the utilities using average criterion?
- Claim: If players use the average criterion $(\sigma_1(t, h(t)), \sigma_2(t, h(t)))$ is a NE
- Intuition: relevant deviation gives a payoff lower than (or equal to) 1

Set of all Nash Equilibrium with Average Criterion

- Denote $C = \{w \mid w$ is in the convex hull of payoff vectors from pure strategy profiles of the stage game$\}$
- Example:



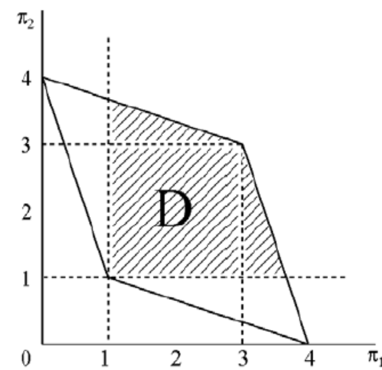
- Can anything in C occur in a NE? No! A players can always assure herself an average payoff of 1 by...

- In general, player i can assure herself an average equal to the minmax payoff,

$$\pi_i^m = \min_{\delta_{-i}} \max_{\delta_i} \pi_i(\delta)$$

by making best response to what the others are supposed to do *in each period* In the worst case, anticipating player i's strategy, the others choose the strategy that minimises her payoff

- Denote $D = \{w \mid w \in C \text{ and } w_i \geq \pi_i^m\}$. Example: in PD $\pi_i^m = 1$ and



Folk Theorem

- Define $E = \{w \mid \text{there is a NE with average payoff vector } w\}$
- Folk Theorem: If players use the average criterion, $E = D$
- Interpretation:
  Almost anything can happen. Comparative statics are problematic
  Even if we cannot write binding contracts, we can obtain the same payoffs through a self-enforcing agreement
  Precise equilibrium may depend on relative bargaining power of parties
  However, we should expect agreements on the efficient frontier
- Other examples: tacit collusion in oligopoly

Sketch of the Proof

- $E \subseteq D$. Already shown.

- Take a sequence of actions $h^*(\infty)$ yielding average payoffs $w \in D$
Show that there exists a (pure strategy) NE for the supergame where $h^*(\infty)$ is taken on the equilibrium path.
Consider the following strategy for player j:
a) in period t, if actual history $h(t)$ conforms with $h^*(t)$ then follow $h^*(\infty)$ and
b) if not, identify the first single-player deviation in $h(t)$ (ignore multi-player deviations)
b.1) denoting the deviating player as i,
if $i \neq j$, then play $\delta_{-i} = \arg\min_{\delta_{-i}} \max_{\delta_i} \pi_i(\delta)$ and if $i = j$ then play an arbitrarily assigned action
b.2) if there is no-single player deviation, then follow $h^*(\infty)$
Following the same arguments as before, both players playing this is a NE

- For all $w \in D$, there exists a sequence of actions $h^*(\infty)$ yielding an average payoff of w

  Idea: alternate actions to produce the same frequency as the randomisation

Nash Equilibria with Discounting Criterion

- Sequence of payoffs evaluated using $u_i(v_i) = \sum_{t=1}^{\infty} \delta^{t-1} v_i(t)$
- Take again strategy (b) above:
  $\sigma_i(t, h(t)) = \begin{array}{l} C \text{ if } t > 1 \text{ and } h(t) \text{ contains a } C \\ DC \text{ otherwise} \end{array}$
- Is (σ_1, σ_2) a NE? If j plays this strategy, then for i the...
  a) Utility from playing this strategy: $\sum_{t=1}^{\infty} \delta^{t-1} 3 = \frac{3}{1-\delta}$ (sequence of 3's)
  b) Utility from deviating: w.l.o.g. assume that i deviates in period 1
  Since j will play C thereafter, the optimal strategy for i is C as well
  Utility: $4 + \sum_{t=2}^{\infty} \delta^{t-1} 1 = 4 + \frac{\delta}{1-\delta}$ (sequence of one 4 and 1's thereafter)
  Hence, best deviation is unprofitable iff $\frac{3}{1-\delta} \geq 4 + \frac{\delta}{1-\delta}$ or $\delta \geq \frac{1}{3}$
- Strategies are a NE as long as players do not discount the future "too much"

Folk Theorem (again)

- It is possible to sustain the other equilibria as long as δ is large
- Need to redefine utility to avoid "unboundedness":
  Redefine: $u_i(v_i) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} v_i(t)$
- Now the set of feasible payoffs is again C
- Interpretation: δ closer to 1 means more equal weighting of periods
- Folk Theorem: For any $w \in D$ with $w_i > \pi_i^m$ for all i, there exists δ^* such that for all $\delta > \delta^*$, w is the payoff vector from some NE
- Proof: similar to that of the other Folk theorem

Subgame Perfect Nash Equilibrium

- NE may not be SPNE. Punishing through minmax may not be credible
- Claim: for PD, all NE above are also SPNE
- Intuition: after a deviation players play a subgame is equal to the original game $\left(\sigma_1^N, \sigma_2^N\right)$ is a NE of the whole game and therefore of these subgames
- More generally, players may use Nash reversion punishments:
  strategies that involve reverting to a stage-game NE play after a deviation
- If these strategies constitute a NE then they also constitute a SPNE
  (valid for all methods of evaluating sequences of payoffs)
- Sometimes, Nash reversion is as severe as minmax punishments
  Examples: PD, Bertrand equilibrium Counterexamples: Cournot
- But we can also obtain Folk theorems for SPNE

Finitely Repeated Games

- Claim: If the stage-game has a unique NE, then there is a unique SPNE of the repeated game, consisting in repeating the stage game equilibrium

- Proof: exercise

- Hence, cooperation cannot be sustained in SPNE

- However, if there are more than one equilibrium, cooperation might be possible

$1\backslash 2$	A	B	C
a	4,4	0,0	0,5
b	0,0	3,3	0,0
c	5,0	0,0	1,1

- NE of the stage game: (b,B) and (c,C). (a,A) is Pareto superior but not a NE
- Suppose two periods with no discounting
- Strategies: Play a (A) in the first period; if outcome was (a,A) play b(B) and otherwise play c(C)
- NE: in any deviation, max gain in the first is 1 and min loss in the second is 2
- SPNE: is a NE in every subgame