Lecture 2: Static Games with Complete Information

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In the Previous Lecture...

- Game Theory: set of tools to analyse behaviour in the presence of *strategic interdependence*
- *Extensive form*: Representation of what players can and cannot do and the consequences (tree structure)
- A *strategy* for a player specifies what to play in each possible circumstance in which the player might be called to play
- For each *strategy profile* there is an outcome and therefore payoffs
- *Normal form*: Representation of the game using strategies
- *Mixed and behavioural* strategies: randomisation over the set of pure strategies and over the set of actions in each information set, resp
- In games of *perfect recall*, both types of randomisation are equivalent

Today's Lecture

- What should we expect players to play?
- Looking for reasonable concepts in simple predictable games and apply these concepts in other settings
- Here, concentrate in simultaneous move games (use normal form).
- Solution concepts:

  Dominant and dominated strategies

  Rationalisable strategies

  Nash equilibrium (reasonable? existence?)
- Assume that rationality (and payoffs) are common knowledge:

Players are rational and all know that the others are rational and all know that the others know that everyone is rational,...

Dominant Strategies

- Example: "prisoner's dilemma". Two arrested individuals answer separately whether they committed a crime. If both confess, sentence of 5 years in prison each. If none confesses, 2 year each. If one confesses and the other does not, 1 and 10 years, respec.

- Representation in normal form:

$1\backslash 2$	DC	C
DC	-2,-2	-10,-1
C	-1,-10	-5,-5

- What will be the outcome? Both confessing! Conflict with Pareto-optimality

- Definition: A strategy $s_i \in S_i$ is a strictly dominant strategy for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$ if for all $s_i' \neq s_i$ we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Dominated Strategies

- Problem: dominant strategies rarely exist. Examples:

(a)

$1\backslash 2$	L	R
U	1,-1	-1,-1
M	-1,1	1,-1
D	-2,5	-3,2

(b)

$1\backslash 2$	L	R
U	5,1	4,0
M	6,0	3,1
D	6,4	4,4

- Definition: A strategy $s_i \in S_i$ is a *strictly* dominated strategy for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$ if there exists $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

- ...it is *weakly* dominated if there exists $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i} \text{ with } > \text{ for some } s_{-i} \in S_{-i}$$

- Examples: D in game (a) is strictly dominated and U and M in (b) are weakly dominated. Should we rule out weakly dominated strategies as well?

Iterated Elimination of Dominated Strategies

- Example: Modified prisoner's dilemma

$1\backslash 2$	DC	C
DC	0,-2	-10,-1
C	-1,-10	-5,-5

- No dominated strategy for Player 1 but DC is dominated for Player 2.
- If Player 2's strategy is eliminated, then DC is dominated for Player 1.
- Iterative process assuming common knowledge of payoffs and of rationality
- Results obtained by eliminating iteratively strictly dominated strategies does not depend on the order of deletion. Deleting *weakly* dominated may. Example: in game (b) in previous slide (U,L,M) leads to (D,R) and (M,R,U) leads to (D,L)
- Extension to mixed strategies (see properties in MWG)

Rationalisable Strategies (Bernheim and Pierce)

- Definition: In a game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$, a strategy σ_i is a best response for player i to her rival's strategy σ_{-i}, $\sigma_i \in b_i(\sigma_{-i})$, if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(S_i)$$

- A strategy σ_i is *never a best response* if there is no σ_{-i} for which σ_i is a best response

- Example: b_4, i.e. (0,0,0,1), is never a best response because (1/2, 0, 1/2, 0) gives always higher payoffs

$1\backslash 2$	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	0,-1

- Definition: In a game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$, the set of strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player i's *rationalisable strategies*
- Example: after eliminating b_4, a_4 can be eliminated. The rest are (iteratively) "rationalisable" $(a_1, b_3, a_3, b_1, a_1...)$ and $(a_2, b_2, a_2, b_2, ...)$:

  Choosing a_1 is resonable because player 1 may believe that player 2 will play b_3

  This belief is reasonable because player 2 may believe that player 1 will play a_3

  This belief is resonable because player 2 may believe that player 1 believes that player 2 will play b_1...

- A strictly dominated strategy is never a best response (but the converse is not necessarily true)
- Eliminating never-best-response strategies leads to smaller set than eliminating strictly dominated strategies
- For 2-player games, however, both processes are equivalent

Nash Equilibrium (Nash 1951)

- Definition: A strategy profile $(s_1, s_2, ..., s_I)$ constitutes a Nash equilibrium of the game $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$ if for every $i = 1, ..., I$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i$$

$1\backslash 2$	l	m	r
U	5,3	0,4	3,5
M	4,0	5,5	4,0
D	3,5	0,4	5,3

- In this example: (M,m) is (the unique) NE.

- In other words, strategy s_i is a best response to s_{-i}. $(s_1, s_2, ..., s_I)$ is a NE iff $s_i \in b_i(s_{-i})$ for all i

- Strategies of the NE profile are rationalisable. Example: (a_2, b_2) in slide 7
- In a NE, players have *correct* beliefs about others' play
- May still not be unique. Example: Coordination game

$1\backslash 2$	ES	GC
ES	100,100	0,0
GC	0,0	1000,1000

- (ES, ES) and (GC, GC) are two pure strategy Nash equilibrium
- Definition: A strategy profile $(\sigma_1, \sigma_2, ..., \sigma_I)$ constitutes a Nash equilibrium of the game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$ if for every $i = 1, ..., I$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(S_i)$$

- See properties in MWG (e.g.: in the MSNE, each player is indifferent among all the pure strategies played with positive probabilty)

A method to Find Pure and Mixed Strategy NE (Example)

1. Find best-response correspondences. For each $[(p, 1-p), (q, 1-q)]$

$$b_1(q) = \begin{cases} 0 \text{ if } q \in [0, \frac{10}{11}) \\ [0,1] \text{ if } q = \frac{10}{11} \\ 1 \text{ if } q \in (\frac{10}{11}, 1] \end{cases} \text{ and } b_2(p) = \begin{cases} 0 \text{ if } p \in [0, \frac{10}{11}) \\ [0,1] \text{ if } p = \frac{10}{11} \\ 1 \text{ if } p \in (\frac{10}{11}, 1] \end{cases}$$

2. Assume that $[(p^*, 1-p^*), (q^*, 1-q^*)]$ is a MSNE and look for conditions:

   If $p^* = 0$ then $q^* = 0$. If $q^* = 0$ then $p^* = 0$. Hence $[(0,1),(0,1)]$ is a NE

   Similarly $[(1,0),(1,0)]$ is a NE.

   If $0 < p^* < 1$ then $q^* = \frac{10}{11}$ and then since $0 < q^* < 1$, $p^* = \frac{10}{11}$. $[(\frac{10}{11}, \frac{1}{11}), (\frac{10}{11}, \frac{1}{11})]$ is a NE.

Existence

- Proposition: Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i()\}]$ in which the sets S_i are finite, has a mixed strategy NE.

- Proposition: A NE in a game $\Gamma_N = [I, \{S_i\}, \{u_i()\}]$ exists if for all $i = 1, ..., I$:

  a) S_i is a non-empty, convex, and compact set of some Euclidean space $\mathbb{R}^M$

  b) $u_i(s_1, ..., s_I)$ is continuous in $(s_1, ..., s_I)$ and quasiconcave in s_i