Lecture 1: Introduction
Motivation and Elements

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November 2005

Introduction: Monopoly Pricing

- In the competitive model, there are many sellers that act as price takers
- In most markets, however, companies can profitably increase prices
- Suppose first that there is only one seller. It solves

$$Max_{q \geq 0} P(q)q - C(q)$$

where $P(\cdot)$ and $C(\cdot)$ are twice-continuous differentiable, $P'(q) < 0$, $P(0) > C'(0)$ and there is a unique q^o such that $P(q^o) = C'(q^o)$

- FOC:

$$P'(q^m)q^m + P(q^m) \leq C'(q^m) \text{ with equality if } q^m > 0$$

- Since $P(0) > C'(0)$ then $q^m \neq 0$ and $P'(q^m)q^m + P(q^m) = C'(q^m)$
- Since $P'(q) < 0$ then $P(q^m) > C'(q^m)$ and $q^m < q^o$

Motivation: Multiple Sellers

- What happens when there are more than one firm (but not many)?
- The demand (and therefore profits) of each firm also depends on the quantity placed (or the price set) by the others
- More generally, the decision-maker well-being depends also on the actions of the others
- Optimal decision depends on the decisions of the others
- Game Theory: framework to analyse decisions in the presence of strategic inter-dependence

Today's Lecture

- Examples and (extensive) representation of a game
- Formal definition of a game
- Concept of strategy
- Normal form representation
- Randomisation

Elements of a "Game"

- Players: who is involved?
- Rules: when do you play?
- Outcomes: for each set of moves, what happens?
- Payoffs: what are the preferences over these outcomes?

Game in Extensive Form

- Example: "Sequential Matching Pennies"

- 2 players: 1 and 2. Player 1 puts first a penny down, heads up or heads down, and then Player 2, after seeing the move of Player 1, puts her penny down, heads up or heads down. If they match, Player 1 pays $1 to Player 2 and if they do not match , Player 2 pays $1 to Player 1.

- Strategic interdependence: a player's payoff is not independent of the actions of the others

- Additional examples

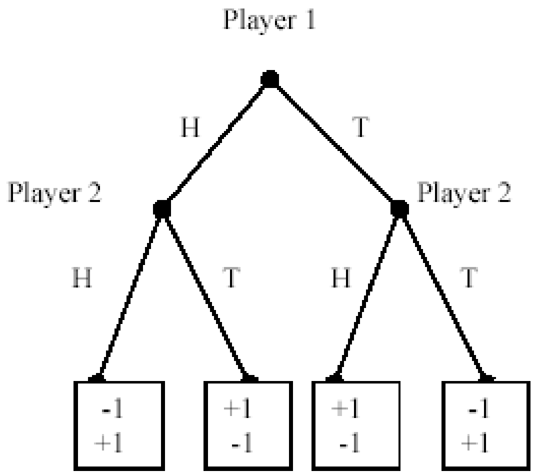


Figure 1:

Simultaneous Moves

- Modified example: suppose that player 1 now hides the penny once put down
- Player 2 does not know where Player 1 played
- Strategically equivalent to "Standard Matching Pennis" (represented equally)
- Game of perfect information: all information sets are singletons
- Game of imperfect information: at least one information set is not a singleton

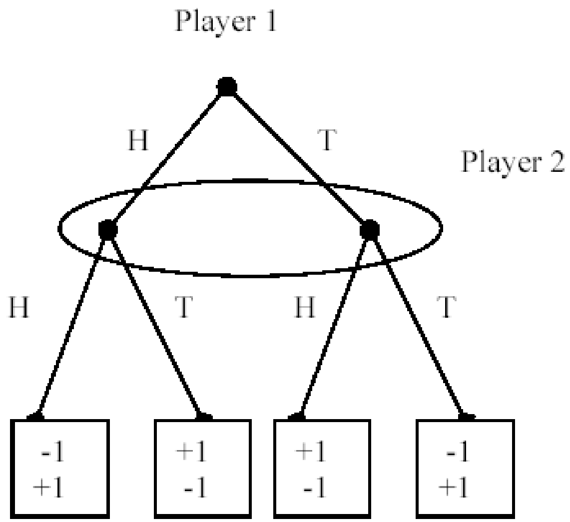
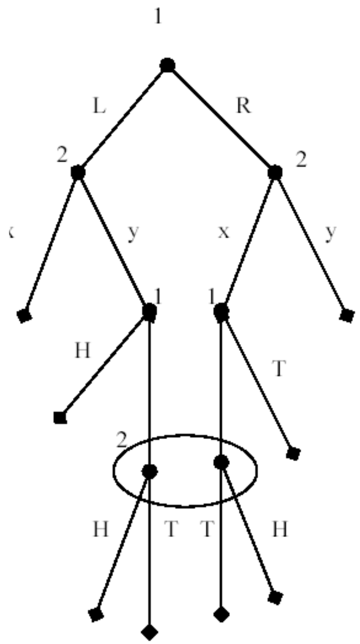


Figure 2:

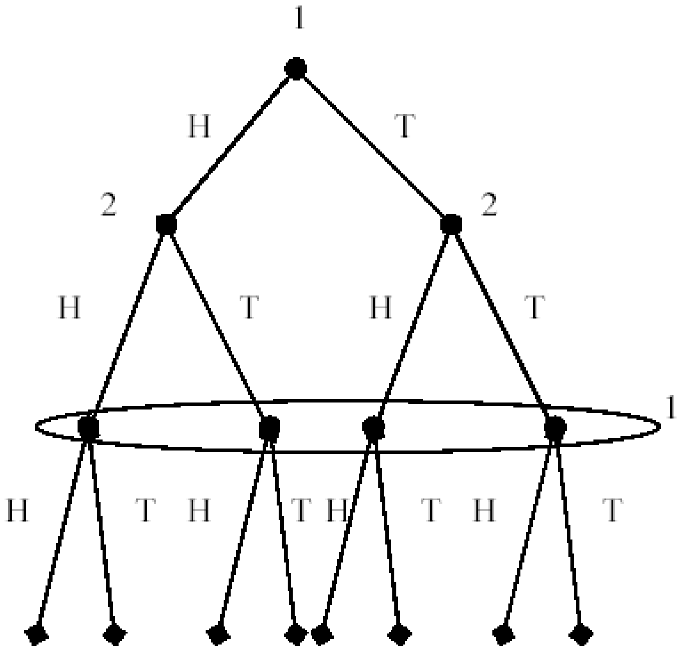
Perfect Recall

- We will assume that players never forget what they knew



Counterexample (1)

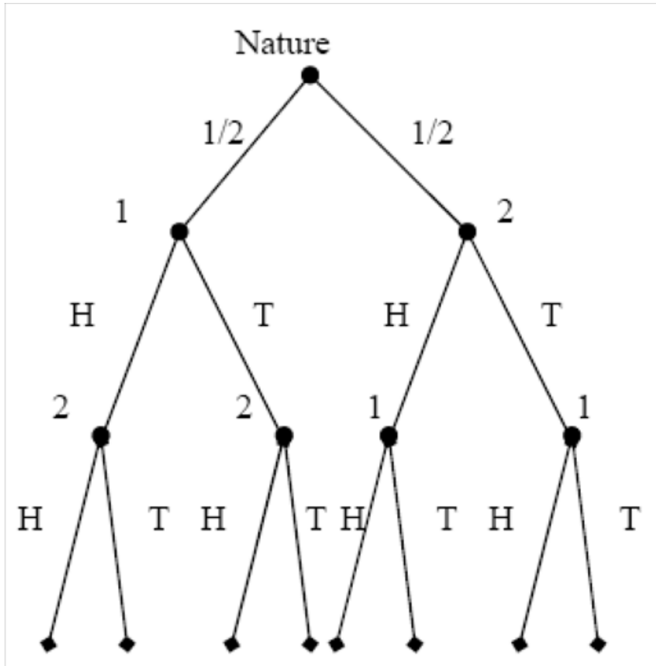
Perfect Recall (2)



Counterexample (2)

Randomness

- We can introduce randomness by adding a "Nature" player
- Nature has probabilities to choose each branch



Formal Definition (Finite Game)

- A game in extensive form can be defined as

$$\Gamma_E = [\chi, \Lambda, I, p(), \alpha(), \Xi, H(), \iota(), \rho(), u] \quad \text{where...}$$

- χ is a finite set of nodes, Λ a finite set of actions, and I a finite set of players
- $p : \chi \longrightarrow \{\chi \cup \varnothing\}$ determines the predecessor, and $p(x) \neq \varnothing$ for all $x \in \chi$ except for x_o, $p(x_o) = \varnothing$

Define $s(x) = \{x' \in \chi; p(x') = x\}$ as the successors of x (the set of successors and predecessors of a node are assumed to be disjoint)

Define $T = \{x \in \chi; s(x) = \varnothing\}$ as the set of terminal nodes

- $\alpha : \chi \backslash \{x_o\} \longrightarrow \Lambda$ determines the action leading to the node (if $x', x'' \in s(x), x' \neq x''$ then $\alpha(x') \neq \alpha(x'')$)

$c(x) = \{a \in \Lambda; a = \alpha(x') \text{ for some } x' \in s(x)\}$ is set of actions available at x

- Ξ is a collection of information sets and $H : \chi \to \Xi$ determines the information set of each node (if $H(x) = H(x')$ then $c(x) = c(x')$)

  $C(H) = \{a \in \Lambda; a \in c(x) \text{ for } x \in H\}$ is the set of actions available at H

- $\iota : H \to \{0, 1, ..., I\}$ determines the player at each info set (0 represents nature)

  $\Xi_i = \{H \in \Xi; \iota(H) = i\}$ is the set of info set at which player i is called to play

- $\rho : \Xi_0 \times \Lambda \to [0, 1]$ assigns probabilities to actions at the nature info sets ($\rho(H, a) = 0$ if $a \notin C(H)$ and $\sum_{a \in C(H)} \rho(H, a) = 1$ for all $H \in \Xi_0$)

- $u = (u_1(\cdot), ..., u_I(\cdot))$ where $u_i : T \to \mathbb{R}$ assigns (Bernoulli) utilities to each terminal node for each player

Extensions

- Continous set of actions $[a, b]$. Examples:
- Infinite set of decision nodes. Examples:
- Infinite number of players. Examples:

Strategy

- Complete contingent plan, specifying how the player will act in every possible situation
- More formally, a strategy for player i is a function

$$s_i : \Xi_i \rightarrow \Lambda \text{ such that } s_i(H) \in C(H) \text{ for all } H \in \Xi_i$$

- Example 1: "Sequential matching pennies":
- Strategies Player 1:
- Strategies Player 2:
- Example 2: "Matching pennies".
- Strategies Player 1: Strategies Player 2:
- Notation: s_{ij} strategy number i player j. $s_{ij} \in S_j$ strategy set of player j
- Strategy profile: $s = (s_1, ..., s_I)$ or $s = (s_i, s_{-i})$, one strategy for each player

Game in Normal Form

- Notice that for each strategy profile s we have a terminal node (or a distribution over the terminal nodes) and therefore payoffs

- Example:

- A game in normal form can be defined as $\Gamma_N = [I, \{S_i\}, \{u_i\}]$, where...

I is the set of players

S_i is the set of strategies for each player ($s_i \in S_i$) and

$u_i(s_1, ..., s_I)$ gives the (von Neumann-Morgensten) utility levels associated with the outcome of $(s_1, ..., s_I)$

- 2-players normal form games can be summarised in matrices

Game in Normal Form: Examples

- Example: "Sequential matching pennies":

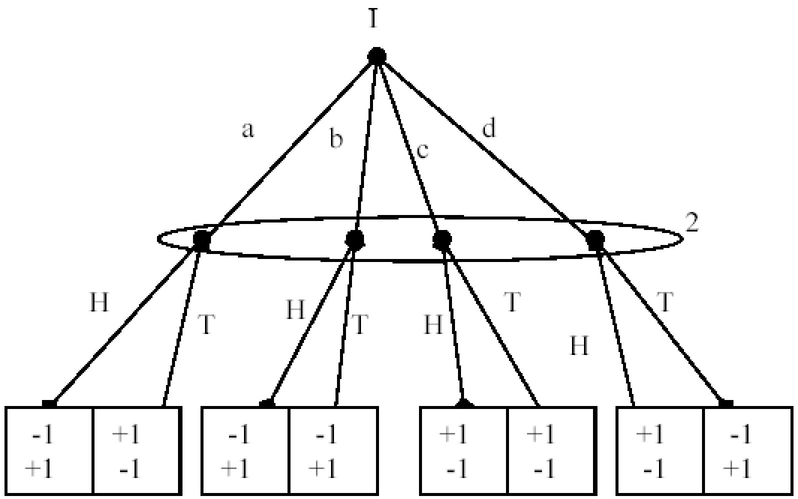
Player1\Player 2	s_{12}	s_{22}	s_{32}	s_{42}
s_{11}	$-1,+1$	$-1,+1$	$+1,-1$	$+1,1$
s_{21}	$+1,1$	$-1,+1$	$+1,-1$	$-1,+1$

- Example: "Matching pennies":

1\2	H	T
H		
T		

From Extensive to Normal Form (and Viceversa)

- For any extensive form representation there is an (essentially) unique normal form representation
- The converse is not true. Example:



Mixed Strategies

- Randomise over actions. Example: government auditing taxpayers
- Notation: $s_i \in S_i$ (determinisitic) *pure* strategy and S_i set of pure strategies.
- Definition: A mixed strategy for player i, $\sigma_i : S_i \rightarrow [0, 1]$, assingns to each pure strategy s_i a probability $\sigma_i(s_i)$ that it will be played (where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$)
- Set of mixed strategies: $\Delta(S_i)$. Representation: simplex. Strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$.
- Given that u_i are of the von Neumann-Morgensten type

$$u_i(\sigma) \equiv E_\sigma[u_i(s)] = \sum_{s \in S_1 x ... x S_I} (\sigma_1(s_1).\sigma_2(s_2)....\sigma_I(s_I))\, u_i(s)$$

- We can redefine again a game in normal form as $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i\}]$

Behavioural Strategies

- In extensive form games, instead of randomising over set of pure strategies, we could randomise at each information set
- Definition: A behavioural strategy for player i assigns to each information set $H \in \Xi_i$ and action $C(H)$, probability $\lambda_i(a, H) \geq 0$ (where $\sum_{a \in C(H)} \lambda_i(a, H) = 1$ for all $H \in \Xi_i$)
- Proposition (Kuhn 1953): In games of perfect recall, both types of randomisation are equivalent
- In games of imperfect recall this is not true. See problem set.

Summary

- Game Theory: set of tools to analyse behaviour in the presence of strategic interdependence
- Extensive form: Representation of what players can and cannot do and the consequences (tree structure)
- A strategy for a player specifies what to play in each possible circumnstance that the player might be called to play
- For each strategy profile (one strategy for each player) there is an outcome and therefore payoffs
- Normal form: Representation using strategies

- Mixed strategies: a player can randomise over the set of deterministic (pure) strategies
- Behavioural strategies: a player can randomise the choice at each information set
- In games of perfect recall, both types of randomisation are equivalent
- Next session: what do people play?